GRAPHS CONTAINING EVERY 2-FACTOR

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Abstract. For a graph G, let $\sigma_2(G) = \min\{d(u)+d(v) : uv \notin E(G)\}$. We prove that every n-vertex graph G with $\sigma_2(G) \geq 4n/3 - 1$ packs with each 2-regular n-vertex graph. This extends a theorem due to Aigner and Brandt and to Alon and Fisher.

1. Introduction

One of the basic results on hamiltonian cycles in graphs, Dirac's theorem [8], says that every n-vertex graph G with minimum degree, $\delta(G)$, at least $n/2$ contains a hamiltonian cycle. The value $n/2$ is best possible. Furthermore, condition $\delta(G) \geq n/2$ does not guarantee that G contains each 2-factor. Corrádi and Hajnal [6] proved that a $3k$-vertex graph G with $\delta(G) \geq 2k$ contains k disjoint triangles. The condition $\delta(G) \geq 2k$ cannot be weakened. Aigner and Brandt [1], and independently Alon and Fisher [2] (for n sufficiently large) extended the Corrádi-Hajnal Theorem as follows.

Theorem 1. *If G is an n-vertex graph with $\delta(G) \geq (2n-1)/3$, then G contains each n-vertex graph H with $\Delta(H) \leq 2$.*

This theorem is also a step towards a conjecture by Bollobás and Eldridge [3], and Catlin [5] on packing of graphs. We will discuss this conjecture and some other graph packing problems in the next section. Fan and Kierstead [10] proved the following strengthening of Theorem 1.

Theorem 2. *If G is an n-vertex graph with $\delta(G) \geq (2n-1)/3$, then G contains the square of a hamiltonian path.*

Ore [16] gave a different sufficient condition for hamiltonicity: he proved that every n-vertex graph G with

$$\sigma_2(G) = \min_{xy \notin E(G)} \{\deg(x) + \deg(y)\} \geq n$$

contains a hamiltonian cycle. Justesen [11] proved an Ore- type version of the Corrádi–Hajnal Theorem by showing that every n-vertex graph G with $\sigma_2(G) \geq 4n/3$ contains $\lfloor n/3 \rfloor$ disjoint triangles. Enomoto [9], and Wang [18] sharpened this result. In particular, they proved the following.

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Theorem 3. *For each positive integer k, every $3k$-vertex graph G with $\sigma_2(G) \geq 4k-1$ contains k disjoint triangles.*

In [15], Theorem 3 was extended as follows.

Theorem 4. *Each n-vertex graph G with $\sigma_2(G) \geq \frac{4n}{3} - 1$ contains all spanning subgraphs whose components are isomorphic to graphs in $\mathcal{H} = \{K_1, K_2, C_3, K_4^-, C_5^+\}$.*

Here K_4^- denotes the graph obtained from the complete 4-vertex graph K_4 by deleting an edge, and C_5^+ is the graph obtained from the 5-cycle C_5 by adding an edge.

The aim of this paper is to prove the following Ore-type analogue of Theorem 1.

Theorem 5. *Each n-vertex graph G with*

$$\sigma_2(G) \geq \frac{4n}{3} - 1 \tag{1}$$

contains every n-vertex graph H with $\Delta(H) \leq 2$.

This theorem is a step toward an Ore-type analogue of the BEC-conjecture discussed in the next session. As it was mentioned above, we will discuss in this section some graph packing problems. Then, in Section 3 we describe the structure of the proof (it will have 6 stages) and give some needed definitions. In Section 4, we state several lemmas that are our main tools for embedding into G of a sequence of subgraphs such that the last subgraph in the sequence is the desired one. In the same section we also prove two of the lemmas that have shorter proofs. The longer proofs are postponed. In Section 5 we show how Stages 2-4 work, and in Section 6 — how Stages 5 and 6 work. In the last three sections, we present the proofs for the lemmas from Section 4.

2. Packings of graphs

Two n-vertex graphs G_1 and G_2 *pack* if there exist injective mappings of their vertex sets onto $[n]$ such that the images of the edge sets do not intersect. Equivalently, G_1 and G_2 *pack* if G_1 is isomorphic to a subgraph of the complement of G_2. This concept leads to a natural generalization of a number of problems in extremal graph theory, such as existence of a fixed subgraph, equitable colorings, and Turán-type problems. In the language of packing, some embedding problems sound more natural. For example, let $\theta(G) = \max\{d(u) + d(v) : uv \in E(G)\}$. Then in the language of packings, the above-mentioned Ore's theorem [16] says that *every n-vertex graph G with $\theta(G) \leq n-2$ packs with the n- cycle C_n*, and our Theorem 5 says that *each n-vertex graph G with $\theta(G) \leq \frac{2n}{3} - 1$ packs with every n-vertex graph H such that* $\Delta(H) \leq 2$. Note that while σ_2 relates to non-adjacent vertices, $\theta(G)$ is a characteristic of edges in G. In [12], this parameter is called *the maximum Ore-degree of G*.

The study of extremal graph packing problems started in the 1970s by Bollobás and Eldridge [3], Sauer and Spencer [17], and Catlin [4]. They considered graph packing under degree constrains. In particular, Bollobás and Eldridge [3], and Catlin [5] stated the following *BEC-conjecture*:

Conjecture 1. *If G_1 and G_2 are n-vertex graphs and $(\Delta(G_1)+1)(\Delta(G_2)+1) \leq n+1$, then G_1 and G_2 pack.*

This is sharp, if true. Theorem 1 above is the case $\Delta(G_2) = 2$ of the BEC-conjecture. Csaba, Shokoufandeh, and Szemerédi [7] also proved the conjecture in the case $\Delta(G_2) \leq 3$ and n is huge, but otherwise, the BEC-conjecture is wide open.

The following Ore-type analogue of the BEC-conjecture was posed in [13].

Conjecture 2. *If G_1 and G_2 are n-vertex graphs and $(0.5\theta(G_1) + 1)(\Delta(G_2) + 1) \leq n + 1$, then G_1 and G_2 pack.*

Thus Theorem 5 verifies the partial case of Conjecture 2 when $\Delta(G_2) = 2$. In fact, we will prove the slightly more general result than Theorem 5, which in the language of packing is as follows.

Theorem 6. *Each n-vertex graph G such that*

$$\theta(G) \leq \frac{2n}{3} - 1 \tag{2}$$

packs with every n-vertex graph H such that $\theta(H) \leq 4$.

3. Proof structure

In this section, we introduce useful notions, and describe the idea of the main proof. We use and somewhat modify the ideas of Aigner and Brandt [1].

Every component of an n-vertex graph H with $\theta(H) \leq 4$, is either a path, or a cycle, or a $K_{1,3}$. We will show the slightly stronger than Theorem 6 statement that every n-vertex graph G satisfying (1), contains each n-vertex graph H whose components are in $\mathcal{F} = \{K_1, K_2, K_4^-\} \cup \{C_{l_j} : 3 \leq l_j \leq n\}$.

A *double i-lasso* (further, simply an *i-lasso*), D_i, consists of a path $x_1, x_2, \cdots, x_i$ with the additional edges x_1x_3 and $x_{i-2}x_i$. For example, $D_4 = K_4^-$.

Here is a big picture of our proof. Let an n-vertex graph G satisfy (1) and let H be an n-vertex graph whose components are in $\mathcal{F}$. We will first embed into G an auxiliary graph H_1 whose every component has at most 5 vertices, namely in $\{K_1, K_2, K_3, K_4^-, C_5^+\}$. Then using this embedding and (1), we will gradually find embedding of graphs whose components are double lassoes which have the same orders as that of the corresponding cycle components of H. Based on these embeddings and Property (1), we will be able to embed H into G. We do this in several stages.

Stage 1. First, for each component R_j of H that is a cycle of length ℓ_j, we represent ℓ_j as the sum of small summands according to the following rules.

(A) If $\ell_j \equiv 0 \pmod 6$, then $\ell_j = 6 + \cdots + 6$.
(B) If $\ell_j \equiv 3 \pmod 6$ and $\ell_j \geq 9$, then $\ell_j = 6 + \cdots + 6 + 3$.
(C) If $\ell_j \equiv 1 \pmod 6$, then $\ell_j = 6 + \cdots + 6 + 3 + 4$.
(D) If $\ell_j \equiv 2 \pmod 6$, then $\ell_j = 6 + \cdots + 6 + 4 + 4$.

(E) If $\ell_j \equiv 4 \pmod 6$ and $\ell_j \geq 10$, then $\ell_j = 6 + \cdots + 6 + 4$.
(F) If $\ell_j \equiv 5 \pmod 6$ and $\ell_j \geq 11$, then $\ell_j = 6 + \cdots + 6 + 4 + 4 + 3$.
(G) If $\ell_j \leq 5$, then $\ell_j = \ell_j$.

Let H_1 be obtained from H by replacing each C_5-component R_j of H with C_5^+ and replacing for each $\ell_j \geq 6$, the component that is the cycle C_{ℓ_j} with the set M_j of disjoint K_3-components and K_4^--components so that to each summand 4 in the above representation of ℓ_j corresponds a K_4^--component, to each summand 3 corresponds a K_3, and to each summand 6 correspond two disjoint K_3s. By construction, H_1 is an n-vertex graph whose every component is in $\mathcal{F}_1 = \{K_1, K_2, K_4^-, K_3, C_5^+\}$. By Theorem 4, G contains a copy of H_1. Graph H_1 will be an *initial H-approximation.*

Stage 2. We start from H-approximation $H' = H_1$ with given sets M_j and will change the approximation and the sets M_j. Given an H-approximation H', a graph H'' is *an H-approximation slightly better than H'* if it is obtained from H' by replacing two K_3-components from the same M_j with a 6-lasso (in both, H' and M_j). From an embedding of H' into G we will obtain an embedding into G of a slightly better graph, if such a graph exists. When the stage ends, we embed into G an H-approximation H_2 that we gradually obtained by slight improvements from H_1 such that each current M_j contains at most one K_3-component. As before, the orders of the components in each M_j sum to ℓ_j.

Stage 3. We start from $H' = H_2$ with given sets M_j. Given an H-approximation H', a graph H'' is *an H-approximation slightly better than H'* if it is obtained from H' by replacing a K_3-component and 6-lasso from the same M_j with a 9-lasso (in both, H' and M_j). From an embedding of H' into G we will obtain an embedding into G of a slightly better H-approximation, if such a graph exists. Let H_3 be the final graph embedded into G in this stage. It has the following structure. If $\ell_j \equiv 0 \pmod 6$, then every component of M_j is a 6-lasso. If $\ell_j \equiv 3 \pmod 6$ and $\ell_j \geq 9$, then one component of M_j is a 9-lasso and all the other are 6-lassoes. If $\ell_j = 7$ then M_j contains one K_3-component and one K_4^--component. If $\ell_j \equiv 1 \pmod 6$ and $\ell_j > 7$, then M_j contains one K_4^--component, one 9-lasso and 6-lassoes. If $\ell_j \equiv 2 \pmod 6$, then M_j contains two K_4^--components and 6-lassoes. If $\ell_j \equiv 4 \pmod 6$ and $\ell_j \geq 10$, then M_j contains one K_4^--component and 6-lassoes. If $\ell_j = 11$ then M_j contains one K_3-component and two K_4^--components. If $\ell_j \equiv 5 \pmod 6$ and $\ell_j \geq 17$, then M_j contains two K_4^--components, one 9-lasso and 6-lassoes.

Stage 4. We start from $H' = H_3$ with given sets M_j. Given an H-approximation H', a graph H'' is *an H-approximation slightly better than H'*, if it is obtained from H' by replacing a K_3-component and K_4^--component from the same M_j with a 7-lasso (in both, H' and M_j). From an embedding of H' into G we will obtain an embedding into G of a slightly better H-approximation, if such a graph exists. Let H_4 be the H-approximation resulting from this stage that is embedded into G.

Stage 5. We start from H' that is obtained from H_4 by replacing each C_5^+-component with a C_5-component. Since $H' \subseteq H_4$, we have an embedding of H' into G. Given an H-approximation H', a graph H'' is *an H-approximation slightly better than H'* if it is obtained from H' by replacing two disjoint lassoes (say of orders z_1 and z_2) from the same M_j with a (z_1+z_2)-lasso. Recall that we view K_4^- as a 4-lasso. As before, from an embedding of H' into G we will obtain an embedding into G of a slightly better H-approximation, if such a graph exists. As the result of this stage, we embed into G the graph H_5 that is obtained from H by replacing each cycle of length $\ell_j \geq 6$ with an ℓ_j-lasso.

Stage 6. We start from the H-approximation $H' = H_5$ and gradually replace each ℓ_j-lasso in H' for $\ell_j \geq 6$ with an ℓ_j-cycle and embed the corresponding graph into G. By construction, the last H-approximation embedded into G will coincide with H.

It is worth to mention that practically repeating our proof of Theorem 5 one can derive the following slightly stronger result.

Theorem 7. *Each n-vertex graph G satisfying (1) contains every n-vertex graph H such that every component of H is either a cycle, or K_1, or K_2, or a double lasso D_ℓ for $\ell \neq 5$.*

4. Basic lemmas

For two subgraphs X and X' of a graph F, let $E_F(X, X')$ denote the set of edges connecting X with X' in F and $e_F(X, X') = |E_F(X, X')|$. For $X \subset V(F)$ and $v \in V(F)$, let $d_F(v, X) = e_F(\{v\}, X)$. If the graph F is clear from the content, we will drop the subscript.

In Stages 5 and 6, an n-vertex graph H'' is an H-*quasi-approximation*, if there exists an H-approximation H' such that H'' is obtained from H' by replacing a C_5-component with a D_5-component. In this case, H' is *slightly better than* H''. A *weak H-approximation* is either an H-approximation or H-quasi-approximation.

From now on, G is an n-vertex graph satisfying (1) with a fixed embedding Ψ of a weak H-approximation H'. When speaking of vertices and subgraphs of H', we usually will mean H' as the subgraph of G defined by Ψ. By definition, in Stages 2–4, the notions of an H-approximation and a weak H-approximation coincide.

Given a pair (G, H') where H' is a weak H-approximation of H embedded into G, a *gadget* is a 4-element vertex set $Y = Y_1 \cup Y_2$ of H', where the 2-element sets Y_1 and Y_2 are chosen as folows. If H' is an H-quasi-approximation, then Y_1 consists of the first two vertices and Y_2 consists of the last two vertices of the only D_5-component in H'. If H' is an H-approximation, then each set Y_i is formed either by the two first (or the two last) vertices of a double lasso in H', or by the two nonadjacent vertices in a K_4^--component in H'. The component of H' containing Y_i will be called the Y_i-*block*, $i = 1, 2$. It may happen that the Y_1-block and the Y_2-block coincide. In this case, Y_1 and Y_2 contain the ends of the same double lasso in H'. By default, we will assume that $Y_1 = \{y_1, y_1'\}$ and $Y_2 = \{y_2, y_2'\}$. Gadgets will help us in Stages 5 and 6 to find an embedding into G of an H-approximation slightly better than H'.

For a gadget Y, a Y-*connector* Y' is a 4-element vertex set obtained from Y either (1) by deleting some $y \in Y_i$ (where $i \in \{1, 2\}$) and adding some y_0 adjacent to $Y_i - y$ and Y_{3-i}, or (2) by deleting some $y \in Y_1$ and $y' \in Y_2$ and adding z and z' such that z is adjacent to $Y_1 - y$ and z', and z' is also adjacent to $Y_2 - y'$. The idea of a Y-connector is the following. If Y_1 and

Y_2 are formed by the first two and the last two vertices in the same double-lasso-component of H' with vertex set D, then for each Y-connector Y', the graph $G[(D-Y)\cup Y']$ contains a cycle with $|D|$ vertices. If the Y_1-block with vertex set V_j and the Y_2-block with vertex set $V_{j'}$ are distinct, then for each Y-connector Y', the graph $G[(V_j \cup V_{j'} - Y)\cup Y']$ contains a double lasso with $|V_j|+|V_{j'}|$ vertices.

For a path $P=(u_1,\dots,u_k)$ and a set $Y\subset V(G)$ and for $2\le i\le k-1$, let $d_3(u_i,Y)=e(\{u_{i-1},u_i,u_{i+1}\},Y)$. The next lemma elaborates a lemma by Aigner and Brandt [1].

Lemma 1. *Let H' be a weak H-approximation embedded into G. Let $Y=Y_1\cup Y_2$ be a gadget. Let $P=(u_1,\dots,u_{\ell+1})$ be a path of length ℓ in H' disjoint from Y. Let $U=\{u_1,\dots,u_{\ell+1}\}$. Assume that the set $Y\cup U$ does not contain a Y-connector Y' such that $G[Y\cup U-Y']$ contains a path of length ℓ from u_1 to u_{l+1}.*

(c1) If $u_i\in U$ is adjacent to $y\in Y_1$ and $y'\in Y_2$, then a vertex $y''\in Y-\{y,y'\}$ cannot be adjacent to all of the neighbors of u_i in P.

(c2) If $d_3(u_i,Y)\ge 9$, then $d(u_i,Y)\le 2$. Furthermore, if $d(u_i,Y)=2$, then u_i cannot have neighbors both in Y_1 and Y_2.

(c3) If some u_i satisfies $d_3(u_i,Y)\ge 9$, then the possible degree sequences of (u_{i-1},u_i,u_{i+1}) in Y are $(4,2,3),(3,2,4),(4,2,4)$ and $(4,1,4)$.

(c4) If $d_3(u_i,Y)+d_3(u_{i+1},Y)\ge 17$ for some i and $d_3(u_i,Y)\ge 9$, then the possible degree sequences in Y for $(u_{i-1},u_i,u_{i+1},u_{i+2})$ are

$$(4,2,3,3),(3,2,4,2),(4,1,4,3),(4,2,4,2),(4,2,4,1).$$

Furthermore, the subgraph of G induced by $Y\cup\{u_{i-1},u_i,u_{i+1},u_{i+2}\}$ is one of the graphs in Figure 1 (up to isomorphism).

Proof of Lemma 1. Statement (c1) is clear, since otherwise $G[U-u_i+y'']$ contains an ℓ-path, and $Y-y''+u_i$ is a Y-connector.

To show (c2), observe that if $d(u_{i-1},Y)+d(u_{i+1},Y)\ge 5$, then u_{i-1} and u_{i+1} have a common neighbor in Y, and if $d(u_{i-1},Y)+d(u_{i+1},Y)\ge 6$, then u_{i-1} and u_{i+1} have at least two common neighbors in Y. So, if $d(u_i,Y)\ge 3$, then we can always find $y\in Y_1$ and $y'\in Y_2$ such that u is adjacent to y and y', and $y''\in Y-\{y,y'\}$ is a common neighbor of u_{i-1} and u_{i+1}, a contradiction to (c1). Furthermore, if $d(u_i,Y)=2$ and u_i has neighbors in both Y_1 and Y_2, then the same argument works.

By (c2), (c3) is clear.

Now we prove (c4). By (c3), (u_{i-1},u_i,u_{i+1}) has one of the four possible degree sequences in Y. In all these sequences, $d(u_{i+1},Y)\ge 3$ and hence $d_3(u_{i+1},Y)\le 8$ by (c2). On the other hand, $d_3(u_{i+1},Y)\ge 17-d_3(u_i,Y)$.

Suppose first that u_i and u_{i+2} have a common neighbor $y\in Y$. We may assume that $y\in Y_1$. Since $U-u_{i+1}+y$ contains a path of length ℓ, u_{i+1} cannot be adjacent to the vertex in Y_1-y. It follows that $d(u_{i+1},Y)=3$. To have $d_3(u_i,Y)\ge 9$, by (c2), we need $d(u_i,Y)=2$ and $d(u_{i-1},Y)=4$. So, we have Case (A) in Figure 1.

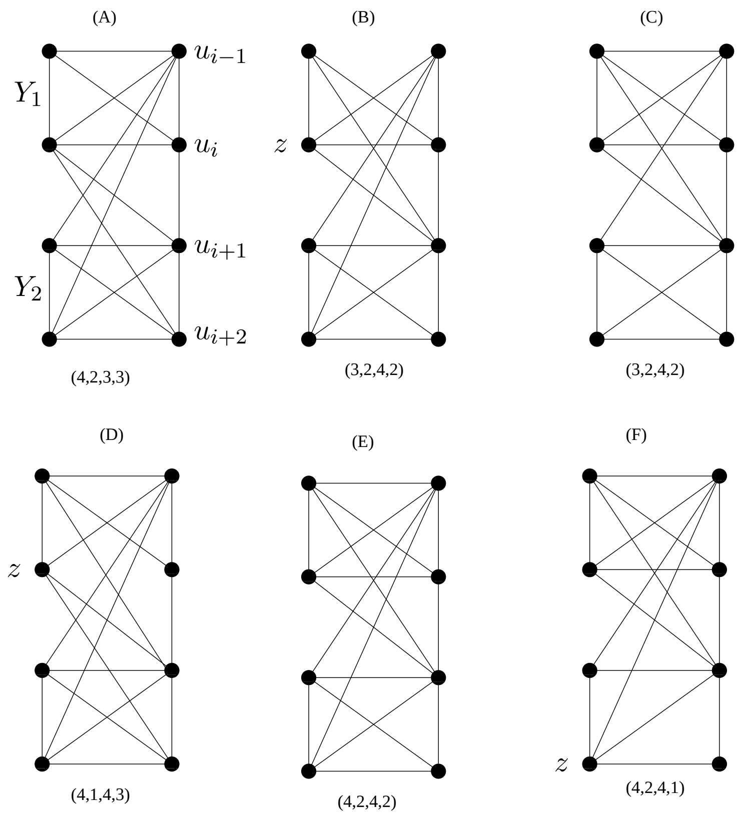

FIGURE 1

Suppose now that u_i and u_{i+2} have no common neighbor $y \in Y$. For this, by (c2), we need $d(u_{i+1}, Y) = 4$. If $d(u_i, Y) = 1$, then to have $d_3(u_i, Y) + d_3(u_{i+1}, Y) \geq 17$, we need $d(u_{i-1}, Y) = 4$ and $d(u_{i+2}, Y) = 3$. This is Case (D) in Figure 1. So, let $d(u_i, Y) = 2$. Since $d(u_{i-1}, Y) \geq 3$, both neighbors of u_i in Y are in the same Y_j, say, in Y_1. If $d(u_{i+2}, Y) = 1$, then $d(u_{i-1}, Y) = 4$ and we have Case (F) in Figure 1. Finally, if $d(u_{i+2}, Y) = 2$, then both neighbors of u_{i+2} in Y are in Y_2 and we have one of Cases (B), (C), or (E) in Figure 1. □

Lemma 2. *Let H' be a weak H-approximation H' in Stage 5 or 6. Let $Y = Y_1 \cup Y_2$ be a gadget and $F \subset V(H')$ be such that $H'[F] = C_k$ with $k \geq 5$ is a component of H' disjoint from Y. If $k \geq 6$ and $e(Y, F) > 8k/3$, then there exists a Y-connector $Y' \subset Y \cup F$ such that $G[(Y \cup F) - Y']$ contains a C_k. If $k = 5$ and $e(Y, F) \geq 14$, then there exists a Y-connector $Y' \subset Y \cup F$ such that $G[(Y \cup F) - Y']$ either contains a C_5 or contains the double lasso D_5. Moreover, when H' is an H-quasi-approximation (and, by definition, Y_1 and Y_2 belong to the only D_5-component of H'), if $k = 5$ and $e(Y, F) \geq 14$, then $G[F \cup F_1]$ contains two disjoint 5-cycles.*

Proof. Let $H'[F] = C_k = (u_1, \dots, u_k)$. Recall that $d_3(u_i, Y) = d(u_{i-1}, Y) + d(u_i, Y) + d(u_{i+1}, Y)$. Since $e(F, Y) > 8k/3$, there exists i such that $d_3(u_i, Y) + d_3(u_{i+1}, Y) \geq 17$. By flipping the order of vertices in F if needed, we may assume that $d_3(u_i, Y) \geq 9$. Lemma 1 yields that the possible degree sequences towards Y of $(u_{i-1}, u_i, u_{i+1}, u_{i+2})$ are

$$(4,2,3,3), (3,2,4,2), (4,1,4,3), (4,2,4,2), (4,2,4,1),$$

and that $G[Y \cup \{u_{i-1}, u_i, u_{i+1}, u_{i+2}\}]$ is one of the graphs in Figure 1 (up to isomorphism).

Recall that by default, $Y_1 = \{y_1, y_1'\}$ and $Y_2 = \{y_2, y_2'\}$. Assume that $d_i = d(y_i, F) \geq d(y_i', F) = d_i'$ for $i = 1, 2$.

CASE 1: $k \geq 6$. Consider configurations (A), (C), and (E) in Figure 1. No matter where y_1 and y_2 are, there is a path from y_1' to y_2' via u_i and u_{i+1}. Furthermore, graph $G[F \cup \{y_1, y_2\} - \{u_i, u_{i+1}\}]$ contains a Hamiltonian path from y_1 to y_2. Observe that $e(\{y_1, y_2\}, F - \{u_i, u_{i+1}\}) \geq d_1 + d_2 - 3 > \frac{4}{3}k - 3 \geq k - 1$. So by the proof of Ore's theorem, this graph is Hamiltonian.

Consider now configurations (B), (D), and (F). If $y_1 = z$ (or $y_2 = z$) in Figure 1, we again obtain a contradiction by the previous argument. Thus we may assume that $z = y_1'$ (or $z = y_2'$) and $d_1 > d_1'$ (or $d_2 > d_2'$). Next, note that if in configuration (B), y_2 were adjacent to u_{i+3}, then the same argument with $\{u_{i+1}, u_{i+2}\}$ in place of $\{u_i, u_{i+1}\}$ would yield a contradiction. So we may assume they are not adjacent, and thus that $d_2 \leq k - 2$ in configuration (B).

We now estimate $d_1 = e(Y, F) - d_1' - d_2 - d_2'$ in (B), (D) and (F). If we replace $\{y_1, y_2\}$ by $\{y_1', y_2'\}$ in the previous argument, then to have $G[F \cup \{y_1', y_2'\} - \{u_i, u_{i+1}\}]$ non-Hamiltonian, by Ore's theorem we will have

$d_1' + d_2' \leq k - 1 + 3$ and $d_2 \leq k - 2$ in (B) and (F),$d_1' + d_2' \leq k - 1 + 2$ and $d_1 \leq k - 1$ in (D).

Set (i, j) equal to $(1, 2)$ in (B) and (F) and equal to $(2, 1)$ in (D). Then

$$d_i > (\frac{8}{3} - 2)k = \frac{2}{3}k \geq \frac{1}{2}k + 1. \tag{3}$$

Now, $Y \cup \{u_i, u_{i+1}\} - \{y_i, y_j'\}$ contains a hamiltonian path from y_i' to y_j and so $G[F \cup \{y_j'\} - \{u_i, u_{i+1}\}]$ is hamiltonian. It follows from (3) that y_i is adjacent to two consecutive vertices of the path $F - \{u_i, u_{i+1}\}$. So, $C_k \subseteq G[F \cup \{y_i, y_j'\} - \{u_i, u_{i+1}\}]$, as desired.

CASE 2: $k = 5$. Since $e(Y, C_5) \geq 14$, we have $d(u_{i-2}, Y) \geq 2$ in graphs (A), (D) and (E), and $d(u_{i-2}, Y) \geq 3$ in graphs (B), (C) and (F). We see that in all cases, except (E), Condition (c1) of Lemma 1 is violated for some subpath of length 3 in our 5-cycle.

In the remaining Case (E), the sequence of degrees in Y for $(u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2})$ is $(2, 4, 2, 4, 2)$, and the configuration is as in Figure 2.

Thus we may partition $G[Y \cup F]$ into a path from Y_1 to Y_2 (non-filled circles in Figure 2) and the lasso D_5 (filled circles in Figure 2).

CASE 3: H' is an H-quasi-approximation (which means that the four vertices of Y are the vertices of degree 2 in a D_5-component F_1 of H') and $k = 5$. Let the fifth vertex of

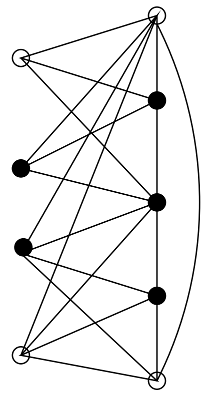

FIGURE 2. The case $(4, 2, 4, 2, 2)$

.

F_1 (adjacent to all vertices in Y) be z. By the proof of Case 2, it is enough to consider the sequence of degrees toward Y for $(u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2})$ equal to $(2, 4, 2, 4, 2)$ and the configuration depicted in Figure 2. If u_{i-2} has a neighbor, say, y_1 in Y_1, then both $G[V(F_1) - y_1 + u_{i-1}]$ and $G[F + y_1 - u_{i-1}]$ contain 5-cycles. Otherwise, $N(u_{i-2}, Y) = Y_2$. Then both $G[\{y_1, y_1', z, u_{i-1}, u_i\}]$ and $G[\{y_2, y_2', u_{i+1}, u_{i+2}, u_{i-2}\}]$ contain 5-cycles. □

The proofs of the next three lemmas will be given in the last three sections.

Lemma 3. *Let H' be a weak H-approximation H' in Stage 5 or 6. Let $Y = Y_1 \cup Y_2$ be a gadget and $F = D_k$ with $k \geq 6$ be a component of H' disjoint from Y. If $e(Y, F) > 8k/3$, then there exists a Y-connector $Y' \subset Y \cup F$ such that $G[(Y \cup F) - Y']$ contains a D_k.*

Lemma 4. *Let H' be a weak H-approximation H' in Stage 5 or 6. Let $Y = Y_1 \cup Y_2$ be a gadget and $F = K_4^-$ be a component of H' disjoint from Y. If $e(Y, F) \geq 11$, then G contains an H-approximation that is slightly better than H'.*

Before stating the last lemma, we need more notions. A *half-gadget* is a set $Z = \{z_1, z_2\} \subset V(H')$ formed either the two non-adjacent vertices of a K_4^--component or by the two first (or last) vertices of a 6-lasso. For a half-gadget $Z = \{z_1, z_2\}$, a *Z-attachment* is a 5-element subset W of $V(G)$ whose vertices can be ordered $w_1, w_2, \ldots, w_5$ so that $w_1 \in Z$, all the edges $w_2w_3, w_3w_4, w_4w_5, w_5w_3$ are in $E(G)$, and either $w_2 \in Z$ or $w_1w_2 \in E(G)$ (see Fig. 3).

We will use such attachments in Stages 3 and 4 to find subgraphs of G that contain 7-lassoes (when the half-gadget is a part of a K_4^--component of H') and 9-lassoes (when the half-gadget is a part of a 6-lasso in H').

Lemma 5. *Let T be the vertex set of a K_3-component of H', and D be the vertex set of a component of H' with $H'[D] \in \{K_1, K_2, K_3, K_4^-, C_5^+, D_6, D_9, D_7\}$ disjoint from T. Let*

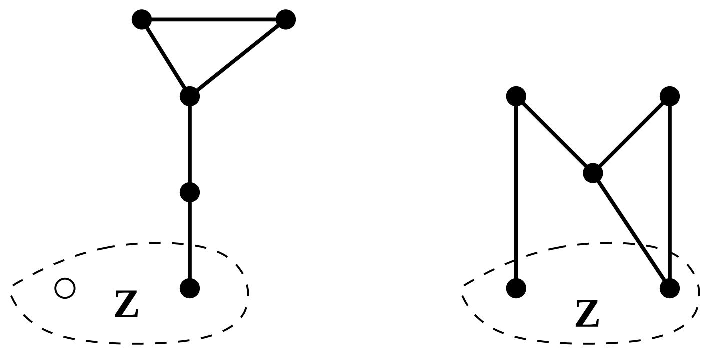

FIGURE 3. Two examples of Z-attachments. The white vertex on the left does not belong to the Z-attachment.

$Z = \{z_1, z_2\} \subset V(H')$ be a half-gadget disjoint from $T \cup D$ and

$$1.5e(Z,D) + e(T,D) > 4|D|. \tag{4}$$

Then $Z \cup T \cup D$ contains a Z-attachment W such that $G[Z \cup T \cup D - W]$ contains $H'[D]$.

5. Embedding small lassoes

Suppose that we have an embedding Ψ into G of an H-approximation H_1 whose components are in $\{K_1, K_2, K_3, K_4^-, C_5^+\}$. In this section we show how Stages 2, 3, and 4 work.

Stage 2: Embedding of 6-lassoes. Suppose that at some step, we have an embedding Ψ into G of an H-approximation H' whose components are K_1's, K_2's, K_3's, K_4^-'s, C_5^+'s, and 6-lassoes but cannot embed into G any slightly better H-approximation. In other words, if H'' is obtained from H' by replacing two K_3-components with a 6-lasso, then H'' is not embeddable into G. Then G has no edges between any two K_3-components of H'.

Let some two K_3-components of H' in the same set M_j have vertex sets $C_1 = \{x_1, x_2, x_3\}$ and $C_1' = \{x_1', x_2', x_3'\}$. By (1),

$$e(C_1 \cup C_1', V(G) - (C_1 \cup C_1')) = \sum_{i=1}^{3}(d(x_i) + d(x_i')) - 12 \geq 3\sigma_2(G) - 12 \geq 4n - 15 > 4(n-6).$$

So there is $D \subset V(H')$ such that $H'[D]$ is a component of H' and $e(C_1 \cup C_1', D) > 4|D|$. If $H'[D] = K_3$, then we are done. So, $H'[D] \in \{K_1, K_2, K_4^-, C_5^+\}$. We will show that we can partition $C_1 \cup C_1' \cup D$ into two subsets W_1 and W_2 so that $G[W_1] \supseteq D_6$ and $G[W_2] \supseteq H'[D]$. That would give an embedding into G of a slightly better H-approximation.

This is easy when $|D| = 1$. If $H'[D] = K_2$, then $e(C_1 \cup C_1', D) \geq 9$. We may assume that $e(C_1, D) \geq 5$. Then there exists $x \in C_1$ adjacent to both vertices in D. Let $X = C_1 - x$ and $Z = C_1' \cup D + x$. Since $e(C_1', D) \geq 9 - 6 > 0$, $G[Z]$ contains D_6.

Let $H'[D] = K_4^-$. Then $e(C_1 \cup C_1', D) \geq 17$. Suppose that a vertex $z \in D$ of degree 2 in $H'[D]$ has at least two neighbors in C_1. In this case, if $D - z$ has a neighbor in C_1', then

$G[C_1' \cup D - z]$ contains a D_6 and $G[C_1 + z]$ contains a K_4^-. Otherwise, $e(D - z, C_1 \cup C_1') \leq 9$ and hence $e(D, C_1 \cup C_1') \leq 15$, a contradiction. So either of the 2-vertices in $H'[D]$ has at most two neighbors in $C_1 \cup C_1'$ and thus $e(D, C_1 \cup C_1') \leq 2 \cdot 2 + 2 \cdot 6 = 16$, a contradiction again.

Let $H'[D]$ be a 6-lasso. If C_1 has a neighbor in one of the triangles of $H'[D]$, then C_1' has no neighbors in the other triangle in $H'[D]$. So $e(D, C_1 \cup C_1') \leq 9 + 9 < 24 = 4|D|$, a contradiction.

Let $H'[D]$ be a 5-cycle $(y_1, y_2, y_3, y_4, y_5)$ with chord y_2y_5. First we prove that

$$e(C_1, \{y_3, y_4\}) \leq 3 \quad \text{and} \quad e(C_1', \{y_3, y_4\}) \leq 3. \tag{5}$$

Indeed, if $e(C_1, \{y_3, y_4\}) \geq 4$, then there is a matching of size two connecting C_1 and $\{y_3, y_4\}$. Thus $G[C_1 \cup \{y_3, y_4\}]$ contains a C_5^+, and hence there is no edge between C_1' and $\{y_1, y_2, y_5\}$. Since $e(C_1 \cup C_1', D) \geq 21$, all other edges between D and $C_1 \cup C_1'$ are present. In particular, $G[C_1' + y_3 + y_4]$ contains a C_5^+ and the subgraph of G on the remaining 6 vertices contain a D_6. Thus, (5) holds.

We may assume that $e(C_1, D) \geq e(C_1', D)$. Then $e(C_1, D) \geq 11$. By (5), $e(C_1, \{y_1, y_2, y_5\}) \geq 8$. Let $x \in C_1$ be either the vertex non-adjacent to y_1 (if exists), or any vertex adjacent to both, y_2 and y_5. Then $G[C_1 - x + y_1]$ is a K_3 and $G[D - y_1 + x]$ contains a C_5^+. In order to avoid a D_6 in $G[C_1' \cup C_1 - x + y_1]$, $e(y_1, C_1') = 0$ and by (5), $e(C_1', D) \leq 6$. On the other hand, also, by (5), $e(C_1, D) \leq 12$ and hence $e(C_1', D) \geq 9$, a contradiction.

Stage 3: Embedding of 9-lassoes. Suppose that at some step, we have an embedding Ψ into G of an H-approximation H' whose components are K_1's, K_2's, K_3's, K_4^-'s, C_5^+'s, and 6- and 9-lassoes but cannot embed into G any slightly better H-approximation. In other words, if H'' is obtained from H' by replacing a K_3-component and a 6-lasso with a 9-lasso, then H'' is not embeddable into G.

Let T be the vertex set of a K_3-component in H' and F be the vertex set of a 6-lasso in H' containing path $(z_1, \ldots, z_6)$ and two chords z_1z_3 and z_4z_6.

Let $Z = \{z_1, z_2\}$. Then $E(\{z_1, z_2\}, T) = \emptyset$. So, by (1), $1.5(d(z_1) + d(z_2)) + \sum_{u \in T} d(u) \geq 3\sigma_2(G) \geq 4n - 3$. Vertices in $T \cup V(F)$ contribute at most $6 + 6$ to $\sum_{u \in T} d(u)$ and at most $2 \cdot 1.5 \cdot 5 = 15$ to $1.5(d(z_1) + d(z_2))$. So, $1.5e(Z, V(G - F) - T) + e(T, V(G - F) - T) \geq 4n - 3 - 12 - 15 > 4(n - 9)$. Thus for some component of H' with vertex set, say D, $1.5e(Z, D) + e(T, D) > 4|D|$. Then Z, T and D satisfy the conditions of Lemma 5. By this lemma, $Z \cup T \cup D$ contains a Z-attachment W such that $G[Z \cup T \cup D - W]$ contains $H'[D]$. Then $G[(F - Z) \cup W]$ contains a 9-lasso.

Stage 4: Embedding of 7-lassoes. Suppose that at some step, we have an embedding Ψ into G of an H-approximation H' whose components are K_1's, K_2's, K_3's, K_4^-'s, C_5^+'s, and 6-, 7-, and 9-lassoes but cannot embed into G any slightly better H-approximation. In other words, if H'' is obtained from H' by replacing a K_3-component and a K_4^--component with a 7-lasso, then H'' is not embeddable into G.

Let T be the vertex set of a K_3-component in H' and F be the vertex set of a K_4^--component in H'. Let $Z = \{z_1, z_2\}$ be the set of degree-2 vertices in $H'[F]$. Then $E(\{z_1, z_2\}, T) = \emptyset$.

As in Stage 3, $1.5(d(z_1)+d(z_2))+\sum_{u\in T} d(u) \geq 4n-3$. Vertices in $T \cup V(D)$ contribute at most $6+6$ to $\sum_{u\in T} d(u)$ and at most $2\cdot 1.5\cdot 3 = 9$ to $1.5(d(z_1)+d(z_2))$. So, $1.5e(Z, V(G-D)-T)+e(T, V(G-D)-T) \geq 4n-3-12-9 > 4(n-7)$. Again, as in Stage 3, for some component of H' with vertex set, say D, $Z\cup T\cup D$ contains a Z-attachment W such that $G[Z\cup T\cup D-W]$ contains $H'[D]$. Then $G[(F-Z)\cup W]$ contains a 7-lasso.

6. Proof of Theorem 5

Recall that the components of the initial H-approximation H' at the beginning of Stage 5 are in the set $\{K_1, K_2, K_3, K_4^-, C_5, D_6, D_7, D_9\}$.

Stage 5: Embedding of ℓ_j-lassoes for all $\ell_j \geq 6$. Let $k = l_j$. If M_j does not consist of a D_k, then it contains some smaller components which are lassoes. We take pairs of components of H' in the same M_j and try to embed into G the graph H'' obtained from H' by replacing such a pair with one bigger double lasso. Suppose that at some step, we cannot proceed. Then by Stage 4, $k\geq 8$ and $k\neq 9$. Recall that the components in M_j now are some double lassoes, and among them at most two K_4^-'s. Choose two such components with vertex sets F_1 and F_2. If $H'[F_i]$ is not a K_4^-, then let Y_i be the set of the two degree-2 vertices in one of the end triangles of $H'[F_i]$. And if $H'[F_i]$ is a K_4^-, then let Y_i be the set of degree-2 vertices in $H'[F_i]$. By the assumption, there are no edges between Y_1 and Y_2. Thus the degree sum of the four vertices in $Y_1\cup Y_2$ is at least $2\sigma_2(G) \geq \frac{8n-6}{3}$.

Suppose first that for an $i \in \{1,2\}$, some $y \in Y_1$ and $y' \in Y_2$ have at least $\frac{4}{3}|F_i| - 1$ neighbors in F_i. Then $H'[F_i]$ is not a K_4^-, and thus is a double lasso D_t for some $t\geq 6$. We may assume that D_t consists of the path $(x_1 = y, x_2, \dots, x_t)$ with edges x_1x_3 and $x_{t-2}x_t$. To avoid a bigger double lasso, y' has no neighbors in $\{x_1, x_2, x_{t-1}, x_t\}$. Since $4t/3-1 > t$, there exists j with $4\leq j\leq t-3$ such that $yx_j, y'x_{j-1} \in E(G)$. So we have a bigger lasso $(H'[F_2]-y', y', x_{j-1}, x_{j-2}, \cdots, y, x_j, \cdots, x_t)$, a contradiction.

If there are no such i, y and y', then $e(Y, V(G)-F_1-F_2) > \frac{8}{3}|V(G)-F_1-F_2|$. So there is $D \subseteq V(G)-F_1-F_2$ such that $H'[D]$ is a component of H' belonging to $\{K_1, K_2, K_3, K_4^-, C_5, D_l : l\geq 6\}$ and $e(Y, D) > \frac{8|D|}{3}$.

If $H'[D] = K_1$ and $D = \{x\}$, then $e(x, Y)\geq 3$. So, x has a neighbor y in Y_1 and a neighbor y' in Y_2. For $y'' \in Y_2 - y'$, graph $G[F_1\cup F_2 + x - y'']$ contains a double lasso with $|F_1\cup F_2|$ vertices.

If $H'[D] = K_2$ and $D = \{x_1, x_2\}$, then $e(\{x_1, x_2\}, Y)\geq 6$. By symmetry, we may assume that $e(\{x_1\}, Y) \geq 3$ and in particular that $Y_1 \subset N_G(x_1)$. If some $y \in Y_1$ is adjacent to x_2, then $G[\{y, x_2\}] = K_2$ and $G[F_1\cup F_2 + x_1 - y]$ contains a double lasso with $|F_1 \cup F_2|$ vertices. If $N(x_2)\cap Y_1 = \emptyset$, then since $e(\{x_1, x_2\}, Y)\geq 6$, we have $Y\subset N_G(x_1)$ and $Y_2 \subset N_G(x_2)$, so we have previous situation with Y_1 and Y_2 switched.

If $H'[D] = K_3$ and $D = X = \{x_1, x_2, x_3\}$, then $e(X, Y) \geq 9$. By symmetry, we may assume that $e(X, Y_1) \geq 5$, $Y_1 = \{y_1, y_1'\}$, $N_G(y_1) \supset X$, and $|N_G(y_1')\cap X| \geq 2$. Since

$e(X,Y_2) \geq 9 - e(X,Y_1) \geq 3$ and $|N_G(y_1') \cap X| \geq 2$, some vertex $x \in N_G(y_1') \cap X$ has a neighbor in Y_2. So, $G[X - x + y_1] = K_3$ and $G[F_1 \cup F_2 + x - y_1']$ contains a double lasso with $|F_1 \cup F_2|$ vertices.

If $H'[D]$ is a K_4^-, then we apply Lemma 4. If $H'[D]$ is D_k with $k \geq 6$, then we apply Lemma 3. Thus we only need to consider the case when $H'[D]$ is a C_5. By Lemma 2, there exists a Y-connector $Y' \subset Y \cup D$ such that $G[(Y \cup D) - Y']$ either contains a C_5 or contains the double lasso D_5. If it contains a C_5, then we are done. So, suppose that G contains the graph H'' obtained from H' by replacing $H'[F_1 \cup F_2 \cup D]$ with a $(|F_1| + |F_2|)$-lasso and a 5-lasso F_3. Suppose that F_3 consists of the path $P = (x_1, x_2, x_3, x_4, x_5)$ plus edges x_1x_3 and x_3x_5. Let $Y_1 = \{x_1, x_2\}$, $Y_2 = \{x_4, x_5\}$ and $Y = Y_1 \cup Y_2$. If at least one edge connecting Y_1 with Y_2 is present in G, then we are done. Otherwise, H'' is an H-quasi-approximation and $\sum_{v \in V(F_3)-x_3} d(v) > 8n/3 - 2 > 8(n-5)/3 + 11$. Since the neighbors in F_3 of these vertices contribute only 8 to this sum, there is a component of H'' with vertex set, say, F_4 that contributes more than $8|F_4|/3$ to this sum.

We now want to show that $Y \cup F_4$ contains a Y-connector Y' such that $G[Y \cup F_4 - Y']$ contains F_4. That would imply that G contains an H-approximation H''' that is slightly better than H'. Repeating the previous argument with the new Y and with F_4 in place of D, we again reduce the problem to the case $F_4 = C_5$. In this case, the last statement of Lemma 2 says that $G[F_3 \cup F_4]$ contains two disjoint 5-cycles. This is what we need.

Stage 6: Embedding of ℓ_j-cycles for all $\ell_j \geq 6$. At the beginning of the stage, every M_j consists of one component, D_{l_j}. Suppose that at some step, we have an embedding into G of an H-approximation H' but cannot embed any slightly better H-approximation. This means that for some $k \geq 6$, a component $H'[F_1]$ of H' is the lasso D_k. Let $H'[F_1]$ consist of the path $P = (y_1, y_2, \ldots, y_k)$ plus edges y_1y_3 and $y_{k-2}y_k$. Let $Y_1 = \{y_1, y_2\}$, $Y_2 = \{y_{k-1}, y_k\}$, and $Y = Y_1 \cup Y_2$. If G contains an edge connecting Y_1 with Y_2, then we are done. Otherwise, we repeat the argument for Stage 5 and $F_1 = F_2$ with one additional possible situation for F: it now can be also a cycle C_ℓ with $\ell \geq 6$. In this case, we apply Lemma 2.

7. Proof of Lemma 3

Assume that $H'[F] = D_k$ consists of a path $(u_1, u_2, \ldots, u_k)$ with the additional edges u_1u_3 and $u_{k-2}u_k$. Let $T_1 = \{u_1, u_2, u_3\}$, $T_2 = \{u_{k-2}, u_{k-1}, u_k\}$, and $P = \{u_4, \ldots, u_{k-3}\}$. Suppose that the lemma is false. We will need the following two claims.

Claim 1. *For $i = 1, 2$, $e(Y, T_i) \leq 8$. Furthermore, if $e(Y, T_i) = 8$, then every vertex in Y has a neighbor in T_i.*

Proof. Suppose that $e(Y, T_1) \geq 9$. Then there exists $i \in \{1, 2\}$ such that $e(Y_i, T_1) \geq 5$, and $e(Y_{3-i}, \{u_1, u_2\}) > 0$. By symmetry, we may assume that $u_1y_{3-i} \in E(G)$. Since $e(Y_i, T_1) \geq 5$, we can rename vertices y_i and y_i' of Y_i so that $u_1y_i \in E(G)$ and (u_2, u_3, y_i') is a triangle. Then (y_i, u_1, y_{3-i}) is a path in G from Y_i to Y_{3-i}, and $G[F - u_1 + y_i']$ contains a k-lasso, a contradiction.

Now suppose that $e(Y,T_1)=8$ and that $y_1\in Y_1$ has no neighbors in T_1. Then the other three vertices of Y have degree sequence $3,3,2$ toward T_1 and one of u_1 and u_2, say u_1, is adjacent to all vertices in $Y-y_1$. Let y_2 be a vertex in Y_2 that has 3 neighbors in T_1. Then $Y-y_2+u_1$ is a Y-connector, and $G[F-u_1+y_2]$ contains a D_k. □

Claim 2. *Let $k\geq 9$. Let $S_1=\sum_{i=1}^3 d(u_i,Y)+\frac{5}{6}d(u_4,Y)+\frac{1}{2}d(u_5,Y)+\frac{1}{6}d(u_6,Y)$ and $S_2=\frac{1}{6}d(u_{k-6},Y)+\frac{1}{2}d(u_{k-5},Y)+\frac{5}{6}d(u_{k-4},Y)+\sum_{i=k-3}^k d(u_i,Y)$. Then $S_1\leq 12$ and $S_2\leq 12$.*

Proof. Assume that $S_1>12$. By Claim 1, $\sum_{i=1}^3 d(u_i,Y)\leq 8$.

CASE 1. $\sum_{i=1}^3 d(u_i,Y)\leq 7$. Then $\frac{5}{6}d(u_4,Y)+\frac{1}{2}d(u_5,Y)+\frac{1}{6}d(u_6,Y)>5$, that is, $5d(u_4,Y)+3d(u_5,Y)+d(u_6,Y)\geq 31$.

If $d(u_4,Y)=4$, then $3d(u_5,Y)+d(u_6,Y)\geq 11$. Since $d(u_i,Y)\leq 4$, (c3) yields that the degree sequence in Y for (u_4,u_5,u_6) is $(4,4,0)$. Then no vertex in Y has a neighbor in T_1 (otherwise, we switch this vertex with u_4), and thus $\sum_{i=1}^3 d(u_i,Y)=0$, and it implies that $S_1=32/5<12$, a contradiction.

If $d(u_4,Y)\leq 3$, then $3d(u_5,Y)+d(u_6,Y)\geq 16$. So $d(u_5,Y)=d(u_6,Y)=4$ and $d(u_4,Y)=3$, a contradiction to (c3).

CASE 2. $\sum_{i=1}^3 d(u_i,Y)=8$. Then $\frac{5}{6}d(u_4,Y)+\frac{1}{2}d(u_5,Y)+\frac{1}{6}d(u_6,Y)>4$, that is, $5d(u_4,Y)+3d(u_5,Y)+d(u_6,Y)\geq 25$.

If $d(u_4,Y)\leq 2$, then $3d(u_5,Y)+d(u_6,Y)\geq 15$. Thus $d(u_5,Y)=4$ and $d(u_6,Y)\geq 3$, a contradiction to (c3).

If $d(u_4,Y)=3$, then $3d(u_5,Y)+d(u_6,Y)\geq 10$. So, (c3) yields $(d(u_5,Y),d(u_6,Y))\in\{(2,4),(3,1),(3,2),(4,0),(4,1)\}$. In any case, since $d(u_4,Y)=3$ and $d(u_5,Y)\geq 2$, there is a neighbor $y\in Y$ of u_5 such that $Y-y+u_4$ is a Y-connector. By Claim 1, y has a neighbor in T_1, and hence we may replace u_4 with y in the double lasso, a contradiction.

If $d(u_4,Y)=4$, then $3d(u_5,Y)+d(u_6,Y)\geq 5$. It follows that $d(u_5,Y)\geq 1$. Again, switch $y\in N(u_5)\cap Y$ with u_4, and we get a contradiction. □

Now we can prove the lemma. By Claim 1,

$$e(Y,P)>\frac{8}{3}|P|=\frac{8}{3}(k-6). \tag{6}$$

It $k=6$, then $P=\emptyset$, and hence $e(Y,P)=0$, a contradiction to (6).

Let $k=7$, i.e., $|P|=1$. Then $e(Y,F)\geq 19$ and $e(Y,u_4)\geq 3$. By symmetry, we may assume that y_1,y_2, and y_1' are neighbors of u_4. Then y_1 and y_1' do not have neighbors in both T_1 and T_2. For the same reason, y_2' does not have neighbors in both T_1 and T_2. Hence there are at least $3\times 3=9$ non-edges between Y and $T_1\cup T_2$. Thus y_2' is a neighbor of u_4. It follows that y_2 does not have neighbors in both T_1 and T_2. We now have a contradiction to $e(F,Y)\geq 19$.

The next case is $k=8$, that is, $|P|=2$. Then $e(Y,F)\geq 22$ and $e(Y,\{u_4,u_5\})\geq 6$. If $e(Y,\{u_4,u_5\})\geq 7$, then u_4 and u_5 have at least three common neighbors in Y, and every common neighbor $y \in Y$ of u_4 and u_5 cannot have neighbors in $T_1 \cup T_2$ (otherwise, we may switch y with u_4 or u_5 to get a Y-connector and a D_8), thus $e(Y, F) \leq 32 - 3 \cdot 6 = 14$, a contradiction. If $e(Y, \{u_4, u_5\}) = 6$, then the common neighbors of u_4 and u_5 in Y have at least 9 non-edges to $T_1 \cup T_2$, thus $e(Y, F) \leq 32 - 9 - 2 = 21$, a contradiction.

If $k = 9$, then by Claim 2, $e(Y, F) \leq 24 = 8/3|F|$, a contradiction.

Now we let $k \geq 10$. Consider the sum

$$S = S_1 + \frac{1}{6}\sum_{i=5}^{k-5}(d_3(u_i, Y) + d_3(u_{i+1}, Y)) + S_2.$$

Observe that $S = e(F, Y) > \frac{8}{3}k$. This and Claim 2 imply that $\sum_{i=5}^{k-5}(d_3(u_i, Y)+d_3(u_{i+1}, Y)) > 6(8/3k - 24) = 16(k - 9)$. Then there exists i with $5 \leq i \leq k - 5$ such that $d_3(u_i, Y) + d_3(u_{i+1}, Y) \geq 17$. By the symmetry between u_i and u_{i+1}, we may assume that $d_3(u_i, Y) \geq 9$. Now statement (c4) of Lemma 1 yields that one of the six configurations in Figure 1 occurs.

Consider P as the union of three paths with the vertex sets $P_1 = \{u_4, \dots, u_{i-1}\}$, $P_0 = \{u_i, u_{i+1}\}$, and $P_2 = \{u_{i+2}, \dots, u_{k-3}\}$.

Claim 3. *Let $k \geq 10$. Suppose that vertices $y_1, y_1' \in Y_1$ and $y_2, y_2' \in Y_2$ are chosen so that*
(R1) $y_1'u_i, y_2'u_{i+1} \in E(G)$ and
(R2) $y_1u_{i-1}, y_2u_{i+2} \in E(G)$.
Then
(S1) for each $5 \leq j \leq i - 1$, if $u_jy_2 \in E(G)$, then $u_{j-1}y_1 \notin E(G)$, and if $u_4y_2 \in E(G)$, then y_1 has no neighbors in T_1;
(S2) similarly, for each $i + 2 \leq j \leq k - 4$, if $u_jy_1 \in E(G)$, then $u_{j+1}y_2 \notin E(G)$, and if $u_{k-3}y_1 \in E(G)$, then y_2 has no neighbors in T_2;
(S3) $d(y_1, P_1) + d(y_2, P_1) + e(Y, T_1) \leq i + 4$ and $d(y_1, P_2) + d(y_2, P_2) + e(Y, T_2) \leq k - i + 4$.

Proof. Suppose first that $u_jy_2 \in E(G)$ and $u_{j-1}y_1 \in E(G)$ for some $5 \leq j \leq i - 1$. Then by (R2), the sequence

$$(T_2, u_{k-4}, u_{k-5}, \dots, u_{i+2}, y_2, u_j, u_{j+1}, \dots, u_{i-1}, y_1, u_{j-1}, u_{j-2}, \dots, u_4, T_1)$$

forms a double lasso of order k in G, and by (R1), $Y - y_1 - y_2 + u_i + u_{i+1}$ is a Y-connector. The same argument proves the second part of (S1), and a symmetric argument proves (S2).

By (S1), $d(y_1, P_1) + d(y_2, P_1) \leq i - 3$. Moreover, the equality is attained only if $i - 4$ is odd and y_1 and y_2 are both adjacent to $u_4, u_6, \dots, u_{i-1}$. Then again by $(S1)$, y_1 is not adjacent to T_1. Therefore, by Claim 1, $e(Y, T_1) \leq 7$. This proves the first part of (S3). Proof of the other part is the same. □

Consider configurations (A), (C) and (E) in Figure 1. For each choice of $y_1 \in Y_1$ and $y_2 \in Y_2$ in these configurations, both (R1) and (R2) hold. So, by Claim 3, (S1) and (S2) hold for each such choice. In particular, if u_4 (respectively, u_{k-3}) has a neighbor in Y_2

(respectively, Y_1), then there are no edges between T_1 and Y_1 (respectively, T_2 and Y_2) which yields $e(Y,T_1) \le 6$ (respectively, $e(Y,T_2) \le 6$). It follows from (S1) and (S2) that

$$e(Y,F) = e(Y,\{u_1,\dots,u_{i-1}\}) + e(Y,\{u_i,u_{i+1}\}) + e(Y,\{u_{i+2},\dots,u_k\}) \le$$
$$(2+2(i-1)) + 6 + (2+2(k-i-1)) = 2k+6.$$

Since $e(Y,F) > 8k/3$, we get $2k+6 > 8k/3$, i.e., $k < 9$, a contradiction.

Consider now configuration (D) in Fig. 1. The set $Y - y_1 - y_2' + u_i + u_{i+1}$ is a Y-connector. Graph $G[F - u_i - u_{i+1} + y_1]$ contains lasso D_{k-1}. So, if y_2' is adjacent to two consecutive vertices in P_1 or P_2, then $G[F - u_i - u_{i+1} + y_1 + y_2']$ contains lasso D_k, a contradiction. Thus $d(y_2',P_1) \le (i-4+1)/2$ and $d(y_2',P_2) \le (k-i-4+1)/2$. Since $y_2'u_i \notin E(G)$, we obtain $d(y_2',P) \le k/2-2$. Since $y_1'u_{i+2} \notin E(G)$, we have $d(y_1',P) \le k-7$. Together with (S3) and the fact that $e(\{y_1,y_2\},P_0) = 2$ we obtain

$$e(Y,F) \le (k/2-2) + (k-7) + (i+4) + (k-i+4) + 2 = 5k/2+1.$$

It follows that $5k/2+1 > 8k/3$, i.e., $k < 6$, a contradiction.

Consider configuration (F) in Fig. 1. The situation here is symmetric to (D). The set $Y - y_1' - y_2 + u_i + u_{i+1}$ is a Y-connector. Graph $G[F - u_i - u_{i+1} + y_2]$ contains lasso D_{k-1}. Vertex y_1' has no two consecutive neighbors on P_1 and P_2. Since $y_1'u_{i+2} \notin E(G)$, $d(y_1',P_2) \le (k-i-4)/2$, $d(y_1',P_1) \le (i-3)/2$, and hence $d(y_1',P) \le (k-3)/2$. Since $y_2'u_i, y_2'u_{i+2} \notin E(G)$, we have $d(y_2',P) \le k-8$. Together with (S3) and the fact that $e(\{y_1,y_2\},P_0) = 3$ we obtain

$$e(Y,F) \le (k-3)/2 + (k-8) + (i+4) + (k-i+4) + 3 = (5k+3)/2,$$

which yields $k < 9$, a contradiction.

Finally, consider configuration (B) in Fig. 1. Again, the set $Y - y_1' - y_2 + u_i + u_{i+1}$ is a Y-connector and $G[F - u_i - u_{i+1} + y_2]$ contains lasso D_{k-1}. Since $y_1'u_{i+2}, y_1'u_{i-1} \notin E(G)$ and y_1' has no two consecutive neighbors on P_1 and P_2, we have $d(y_1',P_1) \le \lfloor (i-4)/2 \rfloor$ and $d(y_1',P_2) \le \lfloor (k-i-4)/2 \rfloor$. Since $y_2'u_i \notin E(G)$, we have $d(y_2',P) \le k-7$. Together with (S3) and the fact that $e(\{y_1,y_2\},P_0) = 3$ we obtain

$$e(Y,F) \le (\lfloor (i-4)/2 \rfloor + \lfloor (k-i-4)/2 \rfloor + 2) + (k-7) + (i+4) + (k-i+4) + 3 \le 5k/2+2, \tag{7}$$

which yields $k < 12$. Furthermore, if $k \in \{10,11\}$, then either $i = 5$ (and hence $\lfloor (i-4)/2 \rfloor = (i-5)/2$) or $k-i = 5$ (and hence $\lfloor (k-i-4)/2 \rfloor = (k-i-5)/2$). In both cases, by (7), $e(Y,F) \le (5k+3)/2$, which yields $k < 9$.

8. Proof of Lemma 4

Let $F = \{w_1,w_2,w_3,w_4\}$ and $H'[F] = K_4^-$ be such that $d_{H'}(w_1) = d_{H'}(w_2) = 2$ and $e(Y,F) \ge 11$. In terms of the complement, this means that

$$e_{\overline{G}}(Y,F) \le 5. \tag{8}$$

Suppose by contradiction that the lemma is not true for Y and F.

Assume first that for some $i \in \{1,2\}$, w_i has neighbors in both, Y_1 and Y_2. By symmetry, we may assume that w_1 is adjacent to y_1 and y_2. If some $y \in \{y_1', y_2'\}$ has at least two neighbors in $\{w_2, w_3, w_4\}$, then $G[\{y, w_2, w_3, w_4\}]$ contains K_4^-, and $Y - y + w_1$ is a Y-connector. So, $e_{\overline{G}}(\{y_1', y_2'\}, \{w_2, w_3, w_4\}) \geq 4$. By (8), w_1 has a neighbor in $\{y_1', y_2'\}$. By symmetry, we may assume that $y_1'w_1 \in E(G)$. Then, as above, y_1 has at most one neighbor in $\{w_2, w_3, w_4\}$. Hence $e_{\overline{G}}(Y, F) \geq 6$, a contradiction. Thus we may assume that w_1 has no neighbors in Y_2, and w_2 has no neighbors in some Y_j, where $j \in \{1,2\}$.

If $j = 2$, then again by (8), w_1 and w_2 have a common neighbor, say y_1, in Y_1. Also by (8), at most one edge between Y_1 and $\{w_3, w_4\}$ is missing. So, by symmetry, we can assume that $y_1w_3, y_1'w_4 \in E(G)$. Then $G[V(F) - w_4 + y_1]$ contains K_4^- and $Y - y_1 + w_4$ is a Y-connector. This contradiction proves that $j = 1$. Furthermore, if $w_1w_2 \in E(G)$, then we can switch the roles of $\{w_1, w_2\}$ and $\{w_3, w_4\}$ thus forcing $e_{\overline{G}}(Y, H) \geq 8$, a contradiction.

So, from now on, $E' = \{w_1w_2, w_1y_2, w_1y_2', w_2y_1, w_2y_1'\} \subseteq E(\overline{G})$. Let $F_1 = F \cup Y$. By (8) and symmetry, we may assume that the only non-edge of $G[F_1]$ that is not in E' (if exists) is either y_1w_1 or y_1w_3 (see Figure 4). So, if the Y_i-block is a K_4^-, then we can switch the roles of F and this block. This implies that

(9) $$\text{neither } Y_1\text{-block nor } Y_2\text{-block is a } K_4^-.$$

Let B_i denote the vertex set of the Y_i-block, $i = 1, 2$.

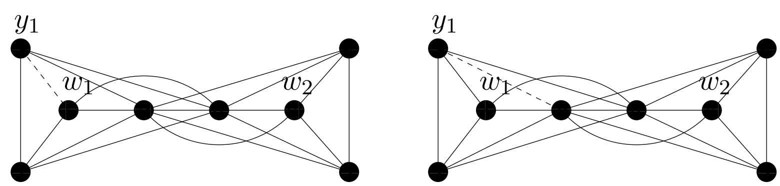

FIGURE 4. The subgraph induced by $F_1 = F \cup Y$.

Let $S = Y \cup \{w_1, w_2\}$. By (2),

$$\sum_{v \in S} d(v) \geq 3\sigma_2(G) \geq 4n - 3.$$

Since the sum gains at most 24 from the neighbors in F_1,

(10) $$e(S, V(G) - F_1) \geq 4n - 27 = 4(n-8) + 5.$$

Therefore, either there exists a component $H'[D]$ of H' such that

(11) $$e(S, D) > 4|D|,$$

or for some $i \in \{1,2\}$ the set $D = B_i - Y_i$ satisfies (11), or $B_1 = B_2$ and the set $D = B_1 - Y$ satisfies (11).

If $H'[D] = K_1$ is a component of H' and $D = \{u\}$, then by (11) there are at least 5 edges from u to S. So, u has a neighbor in Y_1 and a neighbor, say y_2 in Y_2. Then $Y - y_2' + u$ is a Y-connector and $\{y_2'\}$ forms a new K_1-component of H' (see Figure 5).

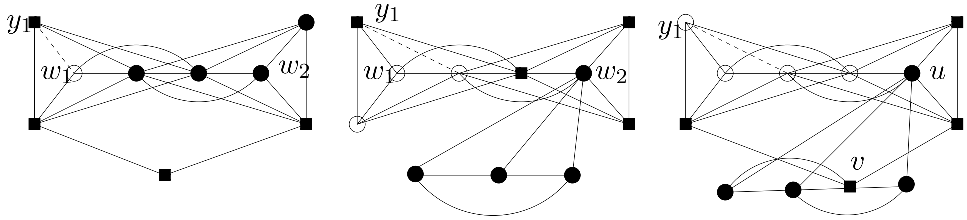

FIGURE 5. Partitions with K_1, K_3 and K_4^-

If $H'[D]$ is a K_2-component of H' and $D = \{u_1, u_2\}$, then by (11), $e(D, S) \geq 9$ and so $e(D, Y) \geq 5$. By symmetry, we may assume that $e(u_1, Y) \geq 3$. We will construct a Y-connector without using F, so we may assume that $N(u_1) \supset Y_1 + y_2$. If u_2 is adjacent to some $y \in Y_1 + y_2$, then $G[\{y, u_2\}] = K_2$ and $Y - y + u_1$ is a Y-connector. Otherwise, since $e(D, Y) \geq 5$, we have $N(u_1) \supset Y$ and u_2 has a neighbor $y \in Y$. So, again $G[\{y, u_2\}] = K_2$ and $Y - y + u_1$ is a Y-connector.

Suppose that $H'[D] = K_3$ is a component of H' and $D = \{u_1, u_2, u_3\}$. By (11), $e(S, D) \geq 13$. If for some $i \in \{1, 2\}$, $|D \cap N(w_i)| \geq 2$, then $G[D + w_i]$ contains a K_4^-, $G[\{y_{3-i}, w_{3-i}, w_3\}] = K_3$, and $Y - y_{3-i} + w_4$ is a Y-connector. Otherwise, $e(Y, D) \geq 13 - 2 = 11$. It follows that $G[D + y_1]$ contains a K_4^-, $\{y_1', w_1, w_3, y_2'\}$ is a Y-connector, and $G[\{w_2, w_4, y_2\}] = K_3$.

Suppose that $H'[D] = K_4^-$ is a component of H' and $D = \{u_1, u_2, u_3, u_4\}$, where $u_1u_2 \notin E(H')$. By (11), $e(Y, D) > 4|D| - e(\{w_1, w_2\}, D) \geq 2|D|$. Then some $u \in D$ has at least 3 neighbors in Y and hence has a neighbor $y \in Y_1$ and a neighbor $y' \in Y_2$. Let $y'' \in Y_2 - y'$. If $e(Y, D) \leq 10$, then by (11), $e(\{w_1, w_2\}, D) \geq 7$, and by symmetry we may assume that $e(w_1, D) = 4$. Then $Y - y'' + u$ is a Y-connector, and each of $G[F - w_1 + y'']$ and $G[D - u + w_1]$ contains K_4^-. So, suppose that $e(Y, D) \geq 11$. Then repeating our argument for F, we may assume that the missing edges in $E(Y, D)$ are u_1y_2, u_1y_2', u_2y_1, u_2y_1', and maybe one more edge. In particular, $e(Y, D) \leq 12$ and hence $e(\{w_1, w_2\}, D) \geq 5$. If $w_2u_1 \in E(G)$, let y be a neighbor of u_1 in Y_1 and $y' \in Y_1 - y$. In this case, $\{y_2, w_2, u_1, y\}$ is a Y-connector, and each of $G[F - w_2 + y_2']$ and $G[D - u_1 + y']$ contains a K_4^-. So, $w_2u_1 \notin E(G)$ and similarly $w_1u_2 \notin E(G)$. Thus, since $e(\{w_1, w_2\}, D) \geq 5$, either $e(w_1, D - u_2) = 3$ or $e(w_2, D - u_1) = 3$. If $e(w_1, D - u_2) = 3$, then let $h \in \{3, 4\}$ be such that $u_hy_1 \in E(G)$ and let $y \in Y_2$ be adjacent to u_2 and $y' \in Y_2 - y$. In this notation, $\{y_1, u_h, u_2, y\}$ is a Y-connector, and each of $G[F - w_1 + y']$ and $G[\{w_1, y_1', u_1, u_{7-h}\}]$ contains a K_4^-. If $e(w_1, D - u_2) < 3$, then $e(w_2, D - u_1) = 3$ and $e(Y, D) = 12$, so that the missing edges in $E(Y, D)$ are only u_1y_2, u_1y_2', u_2y_1, and u_2y_1'. Then $\{y_1, u_1, u_3, y_2\}$ is a Y-connector, and each of $G[F - w_2 + y_1']$ and $G[\{w_2, y_2', u_2, u_4\}]$ contains a K_4^-.

Before considering the remaining cases, we need two facts.

Lemma 6. *Let $P = (u_1, u_2, u_3, u_4)$ be a path in $G - F_1$ and $U = \{u_1, u_2, u_3, u_4\}$. If*

$$d_3(u_2, S) + d_3(u_3, S) \geq 25, \tag{12}$$

then $F_1 \cup U$ can be partitioned into three sets, W_1, W_2, and W_3 so that W_1 is a Y-connector, $G[W_2]$ contains a u_1, u_4-path of length 3, and $G[W_3]$ contains a K_4^-.

Proof. **Case 1:** $d_3(u_2, \{w_1, w_2\}) + d_3(u_3, \{w_1, w_2\}) \geq 9$. We need two claims.

Claim 4. *Let $j \in \{2, 3\}$.*
(a) If $w_1u_{j-1}, w_1u_{j+1}, w_2u_j \in E(G)$, then y_1, y_2, and y_2' are not neighbors of u_j.
(b) If $w_2u_{j-1}, w_2u_{j+1}, w_1u_j \in E(G)$, then y_1', y_2, and y_2' are not neighbors of u_j.

Proof. Let $j = 3$ (the case $j = 2$ is symmetric). Suppose that $w_1u_2, w_1u_4, w_2u_3 \in E(G)$. If $z \in Y_2$ is a neighbor of u_3, then $Y - z + w_4$ is a Y-connector, $G[\{w_3, w_2, u_3, z\}]$ contains a K_4^-, and (u_1, u_2, w_1, u_4) is a path in G. If $y_1u_3 \in E(G)$, then $Y - y_1' - y_2 + u_3 + w_2$ is a Y-connector, $G[\{y_1', w_3, w_4, y_2'\}] = K_4^-$, and (u_1, u_2, w_1, u_4) is a path in G. This proves (a).

Suppose now that $w_2u_2, w_2u_4, w_1u_3 \in E(G)$. Let $w = y_1$ and $w' = y_1'$ if $y_1w_1 \in E(G)$ and $w = y_1'$ and $w' = y_1$ otherwise. If $z \in Y_2$ is a neighbor of u_3 and $z' \in Y_2 - z$, then $Y - w' - z' + u_3 + w_1$ is a Y-connector, $G[\{w', w_3, w_4, z'\}] = K_4^-$, and (u_1, u_2, w_2, u_4) is a path in G. If $y_1'u_3 \in E(G)$, then $Y - y_1' + w_4$ is a Y-connector, $G[\{w_1, w_3, u_3, y_1'\}]$ contains a K_4^-, and (u_1, u_2, w_2, u_4) is a path in G. □

Claim 5. *Let $h \in \{1, 4\}$.*
(a) If $w_1u_2, w_1u_3, w_2u_h \in E(G)$, then either $e(u_{5-h}, Y_2) = 0$ or $e(y_1', \{u_2, u_3\}) = 0$.
(b) If $w_2u_2, w_2u_3, w_1u_h \in E(G)$, then either $u_{5-h}y_1' \notin E(G)$ or $e(Y_2, \{u_2, u_3\}) = 0$.

Proof. By symmetry, we consider only $h = 1$. Assume first that $w_1u_2, w_1u_3, w_2u_1 \in E(G)$, $e(u_4, Y_2) > 0$, and $e(y_1', \{u_2, u_3\}) > 0$. Let z be a neighbor of u_4 in Y_2. Then $Y - y_1' - z + w_3 + w_4$ is a Y-connector, $G[\{y_1', w_1, u_2, u_3\}]$ contains a K_4^-, and (u_1, w_2, z, u_4) is a path in G. This proves (a).

Similarly, assume that $w_2u_2, w_2u_3, w_1u_1 \in E(G)$, $u_4y_1' \in E(G)$, and $e(Y_2, \{u_2, u_3\}) > 0$. Let z be a vertex in Y_2 adjacent to either u_2 or u_3. Then $Y - y_1' - z + w_3 + w_4$ is a Y-connector, $G[\{\{z, w_2, u_2, u_3\}]$ contains a K_4^-, and (u_1, w_1, y_1', u_4) is a path in G. □

Case 1.1: $e(\{u_2, u_3\}, \{w_1, w_2\}) = 3$. To have $d_3(u_2, \{w_1, w_2\}) + d_3(u_3, \{w_1, w_2\}) \geq 9$, we need $e(\{u_1, u_4\}, \{w_1, w_2\}) \geq 3$. Thus there is a matching of size 2 between $\{w_1, w_2\}$ and $\{u_1, u_4\}$, so by symmetry, we may assume that $w_1u_1, w_2u_4 \in E(G)$.

If $w_1u_3, w_2u_2 \in E(G)$, then by Claim 4, y_1, y_2, y_2' are not neighbors of u_2 and u_3, we have a contradiction to $d_3(u_2, S) + d_3(u_3, S) \geq 25$. So we assume that either $w_1u_3 \notin E(G)$ or $w_2u_2 \notin E(G)$ (thus $w_1u_2, w_2u_3 \in E(G)$). If $u_1w_2 \in E(G)$ and $u_4w_1 \in E(G)$, then again by Claim 4, y_1, y_2, y_2' are not neighbors of u_2 and u_3, we have a contradiction to (12). Thus, exactly one of w_1u_3, w_2u_2 is an edge in G and exactly one of u_1w_2, u_4w_1 is an edge in G. So, we have four possibilities.

If $w_2u_1, w_2u_2 \in E(G)$, then by Claim 4 (a), y_1', y_2, y_2' are not neighbors of u_2. So by (12), $e(u_3, Y_1), e(u_3, Y_2) \geq 1$. Let $u_3y_2 \in E(G)$. Then $Y - y_2' + u_3$ is a Y-connector, (u_1, u_2, w_2, u_4) is a path in G, and $G[w_1, w_3, w_4, y_2']$ contains a K_4^-.

Symmetrically, if $w_1u_3, w_1u_4 \in E(G)$, then by Claim 4 (b), y_1, y_2, y_2' are not neighbors of u_3. So by (12), $e(u_2, Y_1), e(u_2, Y_2) \geq 1$, and we may assume that $u_2y_2 \in E(G)$. Then $Y - y_2' + u_2$ is a Y-connector, (u_1, w_1, u_3, u_4) is a path in G, and $G[w_2, w_3, w_4, y_2']$ contains a K_4^-.

If $w_2u_2, w_1u_4 \in E(G)$, then by Claim 4 (b) for $j = 3$, y_1, y_2, y_2' are not neighbors of u_3. By Claim 5 (b) for $h = 4$, either $u_1y_1' \notin E(G)$ or $e(u_2, Y_2) = 0$. So, by (12), $u_1y_1' \notin E(G)$. Now by Claim 5 (b) for $h = 1$, either $u_4y_1' \notin E(G)$, or $e(\{u_2, u_3\}, Y_2) = 0$. So, by (12), $u_4y_1' \notin E(G)$ and all other edges in $E(S, U)$ are present. Thus, $G[y_1', u_2, u_3, w_1]$ contains a K_4^-, $\{y_1, w_3, w_4, y_2\}$ is a Y-connector, and (u_1, y_2', w_2, u_4) is a path in G.

The last possibility is that $w_1u_3, w_2u_1 \in E(G)$. In this case, by Claim 4 (b) for $j = 2$, y_1, y_2, y_2' are not neighbors of u_2. By Claim 5 (a) for $h = 1$, either $e(u_4, Y_2) = 0$ or $e(y_1', \{u_2, u_3\}) = 0$. Since $e(y_1', \{u_2, u_3\}) > 0$ by (12), we conclude that $e(u_4, Y_2) = 0$. Now by Claim 5 (a) for $h = 4$, either $e(u_1, Y_2) = 0$ or $e(y_1', \{u_2, u_3\}) = 0$. Both cases contradict (12).

Case 1.2: $e(\{u_2, u_3\}, \{w_1, w_2\}) = 4$. If both, $e(u_1, \{w_1, w_2\}) > 0$ and $e(u_4, \{w_1, w_2\}) > 0$, then by Claim 4, $e(\{u_2, u_3\}, Y) \leq 2$. This contradicts (12). So, we may assume that $e(u_4, \{w_1, w_2\}) = 0$. Then under the conditions of Case 1, $e(u_1, \{w_1, w_2\}) > 0$.

Case 1.2.1: $u_1w_2 \in E(G)$. By Claim 4, $u_2y_1, u_2y_2, u_2y_2' \notin E(G)$. By Claim 5, either $e(u_4, Y_2) = 0$ or $e(y_1', \{u_2, u_3\}) = 0$. So, by (12), at most one other edge in $E(Y, U)$ is missing and this edge must either be u_1w_2 or be in $E(Y, \{u_1, u_4\})$. If $y_1u_4 \in E(G)$, then $\{y_1', w_1, u_3, y_2'\}$ is a Y-connector, $G[\{w_2, w_3, w_4, y_2\} = K_4$, and (u_1, u_2, y_1, u_4) is a path in G. Otherwise $u_1w_1 \in E(G)$ and hence $\{y_1, u_2, w_2, y_2\}$ is a Y-connector, $G[\{y_1', w_3, w_4, y_2'\}] = K_4^-$, and (u_1, w_1, u_3, u_4) is a path in G.

Case 1.2.2: $u_1w_2 \notin E(G)$. Then $u_1w_1 \in E(G)$. By Claim 4, $u_2y_1', u_2y_2, u_2y_2' \notin E(G)$. By Claim 5, either $u_4y_1' \notin E(G)$ or $e(Y_2, \{u_2, u_3\}) = 0$. So, by (12), $u_4y_1' \notin E(G)$ and at most one other edge in $E(Y, U)$ is missing. Moreover, the missing edge, if exists, must be in $E(Y, \{u_1, u_4\})$. Then u_1 has a neighbor, say z, in Y_2. Therefore, $Y - y_1' - z + w_2 + w_4$ is a Y-connector, $G[\{\{y_1', w_1, w_3, u_2\}]$ contains a K_4^-, and (u_1, z, u_3, u_4) is a path in G.

Case 2: $d_3(u_2, \{w_1, w_2\}) + d_3(u_3, \{w_1, w_2\}) \leq 8$. By (12), $d_3(u_2, Y) + d_3(u_3, Y) \geq 17$. By symmetry, we may assume that $d_3(u_2, Y) \geq 9$. Then by Lemma 1, we only need to consider the 6 configurations in Figure 1 with $i = 2$.

Note that for each configuration in Figure 1, there exist $y \in Y_1, z \in Y_2$ adjacent to both u_1 and u_3. Let $y' \in Y_1 - y$ and $z' \in Y_2 - z$. Then in all cases, $y'u_2, zu_4 \in E(G)$.

If $w_2u_2 \in E(G)$, then $\{y', u_2, w_2, z\}$ is a Y-connector, $G[\{w_1, w_3, w_4, z'\}]$ contains a K_4^-, and (u_1, y, u_3, u_4) is a path in G. Similarly, if $w_1u_2 \in E(G)$, then $\{y_1, w_3, w_2, z'\}$ is a Y-connector, $G[\{w_1, w_3, u_2, y_1'\}]$ contains a K_4^-, and (u_1, z, u_3, u_4) is a path in G. So,

$$w_2u_2 \notin E(G) \quad \text{and} \quad w_1u_2 \notin E(G). \tag{13}$$

Note that in all cases except (E), $d_3(u_2, Y) + d_3(u_3, Y) = 17$ and in (E), $d_3(u_2, Y) + d_3(u_3, Y) = 18$. So, by (12), $d_3(u_2, \{w_1, w_2\}) + d_3(u_3, \{w_1, w_2\})$ is at least 7 in Case (E) and at least 8 otherwise. Thus in the cases other than (E), we do not have other non-edges between $\{w_1, w_2\}$ and $\{u_1, u_2, u_3, u_4\}$. Therefore, $G[\{y, w_1, w_3, w_4\}]$ contains a K_4^-, $\{y', u_2, u_3, z'\}$ is a Y-connector, and (u_1, z, w_2, u_4) is a path in G. This argument also works for (E) if the missing edge is incident with w_1. If not, then w_1 is adjacent to both u_1 and u_4 and y is adjacent to at least one of them. Thus, $\{y', u_2, u_3, z'\}$ is a Y-connector, $G[\{z, w_2, w_3, w_4\}] = K_4$, and either (u_1, y, w_1, u_4) or (u_1, w_1, y, u_4) is a path in G. □

Lemma 7. *Let F_2 be a k-lasso-component of H' that consists of a path $(u_1, \dots, u_k)$ and edges u_1u_3 and $u_{k-2}u_k$, where $k \geq 6$. Denote $T = \{u_1, u_2, u_3\}$. Suppose that $T \cap V(F_1) = \emptyset$ and $e(T, \{w_1, w_2\}) = 6$. If G does not contain an H-approximation slightly better than H', then*
(a) no vertex in $Y - y_1'$ has a neighbor in $\{u_1, u_2\}$;
(b) if u_3 has a neighbor in Y_2, then y_1' has no neighbor in $\{u_1, u_2\}$;
(c) $e(Y, T) \leq 4$.

Proof. Suppose by contradiction that $e(Y_2, \{u_1, u_2\}) > 0$. By symmetry, we may assume that $y_2u_1 \in E(G)$. Define $y = y_1$ if $y_1w_1 \in E(G)$ and let $y = y_1'$ otherwise. Let $y' \in Y_1 - y$. Then $\{y_2, u_1, w_1, y\}$ is a Y-connector, $G[\{y', w_3, w_4, y_2'\}]$ contains a K_4^-, and $G[T - u_1 + w_2]$ contains a K_3. The proof for y_1 in place of y_2 is a bit simpler. This proves (a).

Suppose now that $e(y_1', \{u_1, u_2\}) > 0$ and $e(u_3, Y_2) > 0$. By symmetry, we may assume that $y_1'u_1 \in E(G)$ and $u_3y_2 \in E(G)$. Then $\{y_1, w_3, w_4, y_2'\}$ is a Y-connector, $G[\{y_1', w_1, u_1, u_2\}]$ contains a K_4^-, and $G[T - u_1 - u_2 + w_2 + y_2]$ contains a K_3. This proves (b), and (c) follows from (a) and (b). □

Now we return to the remaining cases of D satisfying (11).

If $H'[D] = C_k = (u_1, \dots, u_k)$, $k \geq 5$, is a component of H', then consider $\sum = \sum_{i=1}^{k}(d_3(u_i, S) + d_3(u_{i+1}, S))$, where indices count modulo k. Then $\sum = 6e(S, D) > 24k$, and so there exists $1 \leq i \leq k$ such that $d_3(u_i, S) + d_3(u_{i+1}, S) > 24$, a contradiction to Lemma 6.

Suppose now that $H'[D]$ is a D_k-component of H' that contains a path $(u_1, \dots, u_k)$ and the edges u_1u_3 and $u_{k-2}u_k$ for some $k \geq 6$. As in the proof of Lemma 3, let $T_1 = \{u_1, u_2, u_3\}$, $T_2 = \{u_{k-2}, u_{k-1}, u_k\}$, and let P denote the path $(u_4, \dots, u_{k-3})$. We may assume that $D \cup Y$ cannot be partitioned into a Y-connector and a set W such that $G[W]$ contains D_k, since otherwise G contains a subgraph slightly better than H'. Thus Claim 1 and Claim 2 hold true.

If $k = 6$, then by Claim 1, $e(Y, T_i) \leq 8$ for $i = 1, 2$. Hence $e(\{w_1, w_2\}, D) > 24 - 8 - 8 = 8$. By symmetry, we may assume that $e(w_1, T_1) = 3$ and $e(w_4, T_2) \geq 2$. Then $G[T_1 + w_1] = K_4$, $G[T_2 \cup \{w_2, w_3, y_2\}]$ contains a D_6, and $Y_1 + w_4 + y_2'$ is a Y-connector.

If $k = 7$, then by Lemma 3, $e(Y, D) \leq 18$, and so $e(\{w_1, w_2\}, D) > 28 - 18 = 10$. If $u_4w_2 \in E(G)$ and w_1 has at least two neighbors in T_i for some $i = 1, 2$, then $G[T_i + w_1]$ contains a K_4^-, $G[T_{3-i} \cup \{u_4, w_2, w_3, y_2\}]$ contains a D_7, and $Y_1 + w_4 + y_2'$ is a Y-connector. Thus if $u_4w_2 \in E(G)$, then $e(w_1, D) \leq 3$ and hence $e(\{w_1, w_2\}, D) \leq 7 + 3 = 10$, a contradiction. A symmetric argument works for w_1 in place of w_2. So we may assume that $u_4w_1, u_4w_2 \notin E(G)$. Then $e(T_1 \cup T_2, \{w_1, w_2\}) \geq 11$, and so for some $i \in \{1, 2\}$, $e(T_i, \{w_1, w_4\}) = 6$. By Lemma 7, $e(T_i, Y) \leq 4$. Then by Claim 1, $e(Y, D) \leq 4 + 8 + 4 = 16$, which in turn gives that $e(T_1 \cup T_2, \{w_1, w_2\}) \geq 29 - 16 = 13$, an impossibility.

If $k = 8$, then by Lemma 3, $e(Y, D) \leq 21$, and so $e(\{w_1, w_2\}, D) > 32 - 21 = 11$. If $e(\{w_1, w_2\}, \{u_4, u_5\}) = 0$, then $e(T_1 \cup T_2, \{w_1, w_2\}) = 12$. Hence by Claim 7, $e(Y, T_1 \cup T_2) \leq 8$, and so $e(Y, D) \leq 8 + 2|Y| = 16$. Thus in this case, $e(\{w_1, w_2\}, D) > 32 - 16 = 16$, an impossibility. If $w_1u_4 \in E(G)$ and w_2 has at least two neighbors in T_1, then $G[T_1 + w_2]$ contains a K_4^-, $Y - y_1' + w_4$ is a Y-connector, and $G[T_2 \cup \{u_5, u_4, y_1', w_1, w_3\}]$ contains a D_8. Repeating this argument with the switched roles of w_1 and w_2 and/or of u_4 and u_5, we conclude that if $e(\{w_1, w_2\}, \{u_4, u_5\}) = j$, then $e(\{w_1, w_2\}, T_1 \cup T_2) \leq 12 - 2j$, and hence $e(\{w_1, w_2\}, D) \leq 12 - j < 12$, a contradiction.

Let $k \geq 9$. As in the proof of Lemma 3, we consider

$$S_1' = \sum_{i=1}^{3} d(u_i, S) + \frac{5}{6}d(u_4, S) + \frac{1}{2}d(u_5, S) + \frac{1}{6}d(u_6, S),$$

$$S_2' = \frac{1}{6}d(u_{k-6}, S) + \frac{1}{2}d(u_{k-5}, S) + \frac{5}{6}d(u_{k-4}, S) + \sum_{i=k-3}^{k} d(u_i, S).$$

$$\text{and} \qquad S' = S_1' + \frac{1}{6}\sum_{i=5}^{k-4}(d_3(u_i, S) + d_3(u_{i+1}, S)) + S_2'$$

Note that these sums are well defined for $k \geq 9$ and that $S' = e(S, D) \geq 4k + 1$. By Lemma 6, $d_3(u_i, S) + d_3(u_{i+1}, S) \leq 24$ for $5 \leq i \leq k - 4$ when $k \geq 9$. Thus $S_1' + S_2' \geq 37$. We may assume that $S_1' \geq 18.5$. By Claim 2, Y contributes at most 12 to S_1'.

If $u_4w_1 \in E(G)$ and $e(w_2, T_1) \geq 2$, then $G[T_1 + w_2]$ contains a K_4^-, $\{y_1, w_4, y_2, y_2'\}$ is a Y-connector, and $G[D - T_1 + \{y_1', w_1, w_3\}]$ contains a D_k. Similar statement holds with the switched roles of w_1 and w_2. So, if $e(\{w_1, w_2\}, u_4) = j$, then $e(\{w_1, w_2\}, T_1) \leq 6 - 2j$, and hence $S_1' \leq 12 + (6 - 2j) + j \cdot \frac{5}{6} + 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{6}$. For $j \geq 1$, this expression is less than 18.5, so $j = 0$. If $e(\{w_1, w_2\}, T_1) \leq 5$, then $S_1' \leq 12 + 5 + \frac{4}{3} < 18.5$. Thus, $e(\{w_1, w_2\}, T_1) = 6$ and by Lemma 7, $e(Y, T_1) \leq 4$. Then

$$S_1' \leq 4 + 4\left(\frac{5}{6} + \frac{1}{2} + \frac{1}{6}\right) + 6 + 2\left(\frac{1}{2} + \frac{1}{6}\right) < 18,$$

a contradiction.

Thus all cases when $H'[D]$ is a component of H' disjoint from F_1 are considered. Now, suppose that the Y_1-block and the Y_2-block is the same component $H'[B]$ of H' and $D =$

$B-Y$. By the definition of a gadget and (9), $H'[B]$ is a double lasso, say $(z_1,\dots,z_k)$, where $z_1=y_1, z_2=y'_1$, $z_{k-1}=y'_2, z_k=y_2$. By (10), since $e(S,V(G)-F_1)\geq 4n-27=4(n-8)+5$ and for every component F_2 of H' distinct from B, $e(S,F_2)\leq 4|F_2|$, we have $e(S,B-Y)\geq 4(|B|-4)+5$. Since $e(S,B-Y)\leq |S|\,|B-Y|=6|B-Y|$, we have $k\geq 7$. As in the proof of the Dirac's theorem, if for some $3\leq i\leq k-4$, $z_iz_k\in E(G)$ and z_1z_{i+1}, then $G[B]$ contains Hamiltonian cycle $(z_1,z_2,\dots,z_i,z_k,z_{k-1},\dots,z_{i+1})$, a contradiction. Since z_1 and z_2 (and z_k and z_{k-1}) are equivalent in B, we conclude that $e(Y,B-Y)\leq 2(1+(k-4))$. It follows that $e(S,B-Y)\leq 2(1+(k-4))+2(k-4)=2+4(k-4)$, a contradiction.

Finally, suppose that $B_1\neq B_2$. By (10), for some $j\in\{1,2\}$,

$$e(S,B_j-Y_j)\geq 3+4(|B_j|-2). \tag{14}$$

Suppose that $H'[B_j]$ is a double lasso $(z_1,\dots,z_k)$, where $\{z_1,z_2\}=\{y_j,y'_j\}$. Similarly to the previous paragraph, if for some $3\leq i\leq k-4$ and some $z\in Y_{3-j}$, $z_iz\in E(G)$ and z_1z_{i+1}, then $G[B_1\cup B_2]$ contains a double lasso that contains B_{3-j} and the path $(z,z_i,z_{i-1},\dots,z_1,z_{i+1},z_{i+2},\dots,z_k)$. By the symmetry between z_2 and z_1, we conclude that $e(Y,B_j-Y_j)\leq 2(1+(k-2))$, a contradiction to (14). This proves the lemma.

9. Proof of Lemma 5

Assume by contradiction that the lemma fails for some choice of Z, T, and D. Everywhere in this section we use notation $T=\{u_1,u_2,u_3\}$. Because of (4), it will be convenient to give to every edge in $E(Z,D)$ weight 1.5 and to every edge in $E(T,D)$ weight 1. Accordingly, for every $A\subseteq D$ and $B\subseteq T\cup Z$, we define $w(A,B)=e(A,T\cap B)+1.5e(A,Z\cap B)$ and $w_{\overline{G}}(A,B)=e_{\overline{G}}(A,T\cap B)+1.5e_{\overline{G}}(A,Z\cap B)$. In these terms, (4) can be rewritten as $w(D,T\cup Z)>4|D|$, or, equivalently,

$$w_{\overline{G}}(D,T\cup Z)<2|D|. \tag{15}$$

If $D=\{x\}$, then by (4), x has a neighbor $z_i\in Z$ and a neighbor in T. So, $D\cup T-z_{3-i}+x$ is a Z-attachment.

Suppose that $D=\{x_1,x_2\}$. By (4), $w(D,T\cup Z)\geq 8.5$. If there is a matching of size 2 connecting D with Z (say, with edges z_1x_1 and z_2x_2), then some vertex of Z (say, x_1) still has a neighbor in T. In this case, $D\cup T-z_2+x_1$ is a Z-attachment and $G[\{x_2,z_2\}]=K_2$. Otherwise, at most two edges connect Z with D and hence all edges connecting T with D are present. Moreover, there is an edge connecting D with Z, say, x_1z_1. Hence for any $c\in T$, $D\cup T-c+x_1$ is a Z-attachment and $G[\{x_2,c\}]=K_2$.

If $H'[D]=K_3$, then D has a neighbor in Z and thus $G[D\cup Z]$ is a Z-attachment.

Let $D=\{x_1,x_2,x_3,x_4\}$ and $H'[D]=K_4^-$ with edge x_2x_4 missing. By (4),

$$w(D,T\cup Z)\geq 16.5. \tag{16}$$

If some $x\in\{x_2,x_4\}$ has at least two neighbors in T, then some other vertex in D still has a neighbor in Z, and hence $D\cup Z-x$ is a Z-attachment. Suppose now that x_2 has exactly one neighbor in T. If x_2 also has a neighbor $z\in Z$, and $z'\in Z-z$, then $T+z+x_2$ is a Z-attachment, and to avoid K_4^- in $G[D-x_2+z']$, z' has two non-neighbors in $D-x_2$. In this case, to satisfy (16), $x_2z' \in E(G)$, but then, as above, z has two non-neighbors in $D-x_2$, a contradiction to (16). So, x_2 has no neighbors in Z. But then, by (16), x_4 has a neighbor in T and a neighbor in Z, a contradiction, as above. Thus our assumption is false and $e(T,\{x_2,x_4\})=0$. It follows that at most one other edge between D and $T\cup Z$ is missing. In particular, we may assume that z_1 is adjacent to all vertices in D and that $z_2x_3 \in E(G)$. Then $Z\cup T-z_1+x_3$ is a Z-attachment and $G[D-x_3+z_1]$ contains a K_4^-.

Let $H'[D]$ be a 6-lasso with triangles T_1 and T_2. If T has a neighbor in T_i and Z has a neighbor in T_{3-i}, then $Z\cup T_{3-i}$ is a Z-attachment, a contradiction. So $w(D,T\cup Z)\le 18 < 24 = 4|D|$, a contradiction to (4).

The remaining three cases, when $H'[D]\in\{C_5^+, D_9, D_7\}$, need much longer proofs.

Case 1: $H'[D]=C_5^+$. Let $H'[D]$ be a 5-cycle (y_1,y_2,y_3,y_4,y_5) with chord y_2y_5. By (4),

$$w(D,T\cup Z)\ge 20.5. \tag{17}$$

We start from a sequence of short claims. Since we assumed that the lemma fails for Z, T, and D, the first statement follows.

(T1) *If $T\cup D$ can be partitioned $T\cup D=W_1\cup W_2$ so that $G[W_1]=K_3$ and $G[W_2]\supseteq C_5^+$, then $e(Z,W_1)=0$.*

(T2) *If $e(D,T)\ge 6$, then for some $y_i,y_j\in D$, $G[T+y_i+y_j]$ contains C_5^+.*

Proof of (T2): Suppose that (T2) fails. Every 5-vertex graph with at least 8 edges contains C_5^+. So, if $e(y_i,T)=3$ for some $y_i\in D$, then $e(y_{i-1},T)=0$, $e(y_{i+1},T)=0$, and $e(y_j,T)\le 1$ for $j=i-2,i+2$. This yields $e(D,T)\le 5$, a contradiction. So, $e(y_i,T)\le 2$ for all $y_i\in D$. Then there are two adjacent y_i and y_j such that $e(y_i,T)=2$ and $e(y_j,T)\ge 1$. For these i and j, $G[T+y_i+y_j]$ contains C_5^+.

(T3) *$e(Z,D)\le 8$ and $e(T,D)\ge 9$.*

Proof of (T3): By (17), $e(D,T)>20-1.5e(D,Z)\ge 5.5$. So $e(T,D)\ge 6$, with equality only if $e(D,Z)=10$. Suppose that $e(D,Z)=10$. In this case, for any 3-vertex subset W of D, $G[W]$ has an edge, and hence the set $Z\cup W$ is a Z-attachment. Furthermore, by (T2) for some two vertices $y_i,y_j\in D$, $G[T+y_i+y_j]$ contains C_5^+. So, the lemma holds in this case. Thus, $e(D,Z)\le 9$ and hence $e(T,D)\ge 7$.

Suppose $e(D,Z)=9$. Then for any $i\in\{1,\dots,5\}$, $Z\cup\{v_{i-1},v_i,v_{i+1}\}$ is a Z-attachment. Since $e(T,D')\ge 7$, for some $j\in\{1,\dots,5\}$, $G[T+y_j+j_{j+1}]$ contains C_5^+. This contradiction shows that $e(Z,D)\le 8$ and hence $e(D,T)>20-1.5e(D,Z)\ge 8$.

(T4) *$E(T,\{y_3,y_4\})$ does not contain a matching of size two. As a result, $e(T,\{y_3,y_4\})\le 3$.*

Proof of (T4): Otherwise, $T\cup\{y_3,y_4\}$ contains a C_5^+. By (T1), there is no edge between Z and the triangle (y_1,y_2,y_5). Then $e(Z,D)\le 4$ and so by (17), $e(T,D)\ge 15$. On the other hand, by (17), $1.5e(Z,D)>20-|T||D|=5$, and hence $e(Z,D)\ge 4$. This means that all edges between Z and $\{y_3,y_4\}$ and between T and D are present. So for any $u\in T$, $G[D-y_3+u]$ contains a C_5^+ and $Z\cup T-u+y_3$ is a Z-attachment.

(T5) *$T \cup D$ has no partition into T' and D' such that $G[T'] = K_3$ and $G[D'] \supset C_5^+$ with $|T \cap T'| \le 1$.*

Proof of (T5): If there is such a partition $T \cup D = T' \cup D'$, then by (T1), $e(Z, T') = 0$. It follows that $e(Z, D) \le 6$, and so $e(T, D) \ge 12$. Since in this case $e(T, \{y_3, y_4\}) \ge 3$, by (T4), we may assume that $e(y_3, T) = 0$. Hence $e(T, D - y_3) = 12$. If $e(Z, \{y_1, y_5\}) > 0$, then for any $u \in T$, $G[T - u + y_2 + y_3 + y_4]$ contains a C_5^+, and $Z + u + y_1 + y_5$ is a Z-attachment, a contradiction. So $e(Z, \{y_1, y_5\}) = 0$, and hence all edges connecting Z and $\{y_2, y_3, y_4\}$ are present. Now $Z + y_2 + y_3 + y_4$ is a Z-attachment, and $G[T + y_1 + y_5]$ contains a C_5^+, a contradiction.

(T6) *If $e(T, \{y_1, y_2, y_5\}) \ge 8$, then $e(T, \{y_3, y_4\}) = 0$.*

Proof of (T6): Assume by contradiction that $e(T, \{y_1, y_2, y_5\}) \ge 8$ and that $u_1 y_4 \in E(G)$. Since at most one edge in $E(T, \{y_1, y_2, y_5\})$ is missing and we can switch the roles of u_2 and u_3, we may assume that $u_3 y_1, u_3 y_5, u_2 y_2 \in E(G)$. So, if $u_1 y_2 \in E(G)$, then $G[\{u_3, y_1, y_5\}] = K_3$, and $G[\{u_1, u_2, y_2, y_3, y_4\}]$ contains a C_5^+, a contradiction to (T5). Thus $u_1 y_2 \notin E(G)$ and all other edges connecting T with $\{y_1, y_2, y_5\}$ are present. Therefore, if $y_4 u_j \in E(G)$ for some $j \in \{2, 3\}$, then we switch the roles of u_1 and u_j and the previous argument works. So, $e(y_4, T) = 1$. Furthermore, if y_3 has a neighbor $u_i \in T$, then we can switch the roles of y_3 and y_4 and the roles of y_2 and y_5: since $e(v_5, T) = 3$, our argument works. Thus the last possibility is that $E(T, \{y_3, y_4\}) = \{y_4 u_1\}$. Then $e(T, D) = 9$ and so $e(Z, T) \ge 8$. On the other hand, since $G[\{y_1, u_2, u_3\}] = K_3$ and $G[D - y_1 + u_1]$ contains a C_5^+, by (T1), $e(y_1, Z) = 0$. It follows that $e(Z, D - y_1) = 8$, and hence $T + y_4 + z_1$ is a Z-attachment, and $G[D - y_4 + z_2]$ contains a C_5^+.

(T7) *$e(T, D) \le 10$ and $e(Z, D) \ge 7$.*

Proof of (T7): By symmetry, assume that $e(y_4, T) \ge e(y_3, T)$. If $e(T, D) \ge 11$, then by (T4), $e(T, \{y_1, y_2, y_5\}) \ge 8$, and so by (T6), $e(T, \{y_3, y_4\}) = 0$, a contradiction to $e(T, D) \ge 11$.

(T8) *$6 \le e(T, \{y_1, y_2, y_5\}) \le 7$.*

Proof of (T8): By (T3) and (T4), $e(T, \{y_1, y_2, y_5\}) \ge 6$. Suppose that $e(T, \{y_1, y_2, y_5\}) \ge 8$. Then by (T6), $e(T, \{y_3, y_4\}) = 0$. By (T3), $e(T, \{y_1, y_2, y_5\}) = 9$ and $e(Z, D) = 8$. Since $G[T - u_1 + y_1] = K_3$ and $G[D - y_1 + u_1]$ contains a C_5^+, (T1) implies that $e(Z, D - y_1) = 8$, and hence $Z \cup \{y_2, y_3, y_4\}$ is a Z-attachment, and $G[T + y_1 + y_5] = K_5$, a contradiction.

(T9) *$e(T, \{y_3, y_4\}) \le 2$. So $e(Z, D) = 8$, $e(T, \{y_3, y_4\}) = 2$, and $e(T, \{y_1, y_2, y_5\}) = 7$.*

Proof of (T9): If $e(T, \{y_3, y_4\}) \ge 3$, then by (T4), $e(T, \{y_3, y_4\}) = 3$ and exactly one of y_3 and y_4 (we may assume y_3) is adjacent to all vertices of T. Suppose first that y_5 has no neighbors in T. Then by (T8), $e(\{y_1, y_2\}, T) = 6$ and hence $G[T + y_1 - u_1] = K_3$ and $G[D - y_1 + u_1]$ contains a C_5^+. By (T0), this yields that $e(y_1, Z) = 0$. On the other hand, since $e(T, D) = 9$, by (17), we have $e(Z, D) \ge 8$, and so $e(Z, D - y_1) = 8$. In this case, $Z \cup \{y_5, y_3, y_4\}$ is a Z-attachment, and $G[T + y_1 + y_2] = K_5$, a contradiction.

So by symmetry we may assume that $u_1y_5 \in E(G)$. Then for $j = 2, 3$, $G[\{y_3, y_4, y_5 + u_1 + u_j\}]$ contains a C_5^+, and hence by (T5), $G[\{u_2, y_1, y_2\}]$ and $G[\{u_3, y_1, y_2\}]$ are not triangles. Thus $e(T - u_1, \{y_1, y_2\}) \leq 2$. So by (T8), $e(y_5, T) \geq 2$, and we may assume that $u_2y_5 \in E(G)$. Repeating the argument with u_2 in place of u_1, we conclude that $e(u_1, \{y_1, y_2\}) \leq 1$. So, by (T8), $e(y_5, T) = 3$. It follows that $G[\{u_1, y_1, y_2\}] = K_3$ and $G[\{y_3, y_4, y_5, u_2, u_3\}]$ contains a C_5^+, a contradiction to (T5).

We now are ready to finish Case 1. If for some $1 \leq i \leq 3$, $e(u_i, \{y_3, y_4\}) = 2$, then $G[\{u_i, y_3, y_4\}] = K_3$ and $G[\{y_1, y_2, y_5, u_3\} \cup (T - u_i)]$ contains a C_5^+, a contradiction to (T5). Thus by (T9) and (T4), we may assume that $y_3u_1, y_3u_2 \in E(G)$. If $y_5u_1, y_5u_2 \notin E(G)$, then by (T9), all other edges connecting T and $\{y_1, y_2, y_5\}$ are present. Hence $G[\{u_2, u_3, y_1\}] = K_3$ and $G[D - y_1 + u_1]$ contains a C_5^+. So by (T0), $e(y_1, Z) = 0$ and hence by (T9), $e(Z, D - y_1) = 8$. It follows that $G[Z + y_3 + y_4 + y_5]$ is a Z-attachment, and $G[T + y_1 + y_2]$ contains a C_5^+, a contradiction. Thus assume $u_1y_5 \in E(G)$.

Now $G[u_1, u_2, y_3, y_4, y_5]$ contains a C_5^+. Then by (T5), $G[\{u_3, y_1, y_2\}] \neq K_3$, and hence for some $i \in \{1, 2\}$, $u_3y_i \notin E(G)$.

Since $u_3y_i \notin E(G)$, by (T9), at most one edge is missing in $E(T - u_3, \{y_1, y_2, y_5\})$. So, by the symmetry between u_1 and u_2, we may assume that $u_1y_1, u_1y_2, u_2y_5 \in E(G)$. Then $G[\{u_1, y_1, y_2\}] = K_3$ and $G[u_2, u_3, y_3, y_4, y_5]$ contains a C_5^+, a contradiction to (T5). This finishes Case 1.

Case 2: $H'[D] = D_9$. Assume that $H'[D]$ contains a path $(x_1, \ldots, x_9)$ and edges x_1x_3 and x_7x_9. Let $T_1 = \{x_1, x_2, x_3\}$ and $T_2 = \{x_7, x_8, x_9\}$. By (15),

$$w_{\overline{G}}(D, T \cup Z) < 18. \qquad (18)$$

Claim 6. *For $i = 1, 2$, $w_{\overline{G}}(T_i, Z \cup T) \geq 3$.*

Proof. Suppose that $w_{\overline{G}}(T_1, Z \cup T) \leq 2.5$. Then $e(Z, T_1) \geq 5$ and $e(T_1, T) \geq 7$. Thus by symmetry we may assume that $e(z_1, T_1) = 3$ and $e(z_2, T_1) \geq 2$. Let $x \in \{x_1, x_2\}$ be a neighbor of z_2. Since $e(T_i, T) \geq 7$, x has a neighbor in T. Then $T + z_2 + x$ is a Z-attachment, and $G[D - x + z_1]$ contains a D_9, a contradiction. □

Claim 7. *If $e(Z, T_1) > 0$ (respectively, $e(Z, T_2) > 0$), then $e(x_4, T) = 0$ (respectively, $e(x_6, T) = 0$).*

Proof. If $e(Z, T_1) > 0$ and $e(x_4, T) > 0$, then $G[D - T_1 + T]$ contains a D_9 and $Z \cup T_1$ is a Z-attachment. □

Assume first that $e(Z, T_2) = 0$, i.e., $w_{\overline{G}}(Z, T_2) = 9$. Then by (18), $w_{\overline{G}}(Z, T_1) < 18 - 9 = 9$, i.e., $e(Z, T_1) > 0$. So by Claim 7, $e(x_4, T) = 0$, and hence by Claim 6,

$$w_{\overline{G}}(D, T \cup Z) - w_{\overline{G}}(Z, T_2) - w_{\overline{G}}(T_1, Z \cup T) - w_{\overline{G}}(x_4, T) < 18 - 9 - 3 - 3 = 3.$$

In particular, $e(x_5, T) > 0$ and $e(\{x_4, x_5, x_6\}, Z) \geq 8$. Since at most one edge is missing in $E(\{x_4, x_5, x_6\}, Z)$, by the symmetry between z_1 and z_2, we may assume that $x_5z_1, x_4z_2, x_6z_2 \in E(G)$. Then $G[D - x_5 + z_2]$ contains a D_9 and $T + z_j + x_2$ is a Z-attachment, a contradiction.

So we have $e(Z, T_i) > 0$ for $i = 1, 2$. By Claim 7, x_4 and x_6 have no neighbors in T.

Claim 8. $w_{\overline{G}}(T_1 \cup T_2, Z \cup T) \geq 9$.

Proof. Suppose that

$$w_{\overline{G}}(T_1 \cup T_2, Z \cup T) < 9. \tag{19}$$

Assume first that $e(Z, T_1) \geq 5$. Then for $i = 1, 2$, there exists $j = j(i) \in \{1, 2\}$ such that $x_iz_j \in E(G)$ and $G[T_1 - x_i + z_{3-j}] = K_3$. So for $i = 1, 2$, if x_i has a neighbor in T, then $T + z_j + x$ is a Z-attachment, and $G[D - x_i + z_{3-j}]$ contains a D_9, a contradiction. Therefore, $e(\{x_1, x_2\}, T) = 0$. Hence by (19), $w_{\overline{G}}(T_2, Z \cup T) < 3$, a contradiction to Claim 6. So, by the symmetry between T_1 and T_2, we conclude that $e(Z, T_i) \leq 4$ for $i = 1, 2$.

Since $e_{\overline{G}}(T_1 \cup T_2, Z) \geq 4$, by (19) we have $w_{\overline{G}}(T_1 \cup T_2, T) < 3$. Thus by the symmetry between T_1 and T_2, we may assume that $e(T, T_1) \geq 8$. If for some $i \in \{1, 2\}$, $e(x_i, Z) > 0$, then we can choose some $u \in T$ such that $G[T - u + x_i] = K_3$ and $G[T_1 + u - x_i] = K_3$. In this case, $Z \cup (T - u) + x_i$ is a Z-attachment, and $G[D - x_i + u]$ contains a D_9. We conclude that $e(Z, \{x_1, x_2\}) = 0$ and hence $w_{\overline{G}}(T_1, Z) \geq 6$. This together with (19) contradicts Claim 6. □

Claim 8 implies that

$$w_{\overline{G}}(\{x_4, x_5, x_6\}, T \cup Z) - w_{\overline{G}}(\{x_4, x_6\}, T) < 18 - 9 - 6 = 3.$$

So $e(x_5, T) > 0$ and $e(\{x_4, x_5, x_6\}, Z) \geq 5$. Thus we may assume that $e(z_1, \{x_4, x_5, x_6\}) = 3$ and $e(z_2, \{x_4, x_5, x_6\}) \geq 2$. By symmetry we may assume that $x_4z_2 \in E(G)$. Then $T_1 + x_4 + z_2$ is a Z-attachment, and $G[T_2 \cup T \cup \{x_5, x_6, z_1\}]$ contains a D_9, a contradiction.

Case 3: $H'[D] = D_7$. Assume that $H'[D]$ contains a path $(x_1, \ldots, x_7)$ and edges x_1x_3 and x_5x_7. Let $T_1 = \{x_1, x_2, x_3\}$ and $T_2 = \{x_5, x_6, x_7\}$. By (15),

$$w_{\overline{G}}(D, T \cup Z) < 14. \tag{20}$$

If $e(x_4, T) > 0$ and $e(Z, T_1) > 0$, then $G[D - T_1 + T]$ contains a D_7 and $Z \cup T_1$ is a Z-attachment. So by symmetry, if $e(x_4, T) > 0$, then $e(Z, T_1 \cup T_2) = 0$, and hence $w_{\overline{G}}(D, T \cup Z) \geq 1.5 \cdot 12 = 18$, a contradiction to (20). So,

$$e(x_4, T) = 0. \tag{21}$$

Claim 9. *For $i = 1, 2$, $w_{\overline{G}}(T_i, T \cup Z) \geq 5$, and if $w_{\overline{G}}(T_i, T \cup Z) = 5$, then $e(Z, T_i) = 4$ and $e(T, T_i) = 7$.*

Proof. Assume that $w_{\overline{G}}(T_1, T \cup Z) \le 5$. Then $e(Z, \{x_1, x_2\}) > 0$, that is, for some $j \in \{1, 2\}$, x_j has a neighbor in Z. If $e(T_1, T) \ge 8$, then since at most one edge in $E(T_1, T)$ is missing, we may assume that $x_j u_1, x_j u_2, x_{3-j} u_3, x_3 u_3 \in E(G)$. In this case, $G[D - x_j + u_3]$ contains a D_7, and $Z \cup T - u_3 + x_j$ is a Z-attachment. So, $e(T_1, T) \le 7$, i.e.,

$$w_{\overline{G}}(T_1, T) \ge 2. \tag{22}$$

Suppose now that $e(T_1, Z) \ge 5$. Since $w_{\overline{G}}(T_1, T) \le 5$, for some $j \in \{1, 2\}$, x_j has a neighbor in T. Since at most one edge is missing in $E(T_1, Z)$, we may assume that $x_j z_1, x_{3-j} z_2, x_3 z_2 \in E(G)$. Then $G[D - x_j + z_2]$ contains a D_7, and $T + z_1 + x_j$ is a Z-attachment. Thus $e(T_1, Z) \le 4$, i.e., $w_{\overline{G}}(T_1, Z) \ge 3$. This together with (22) yields the claim. □

By (20) and (21), $w_{\overline{G}}(T_1, T \cup Z) + w_{\overline{G}}(T_2, T \cup Z) \le 11$. So by Claim 9, we may assume that $w_{\overline{G}}(T_1, T \cup Z) = 5$, and therefore $e(Z, T_1) = 4$ and $e(T, T_1) = 7$. Also by (20), (21), and Claim 9,

$$w_{\overline{G}}(x_4, Z) < 14 - w_{\overline{G}}(x, T) - w_{\overline{G}}(T_1 \cup T_2, T \cup Z) \le 14 - 3 - 10 = 1,$$

which means that $x_4 z_1, x_4 z_2 \in E(G)$. Since $e(Z, T_1) = 4$, either z_1 or z_2 (say z_1 by symmetry) has at least two neighbors in T_1. Since $e(T, T_1) = 7$, every vertex in T_1 has a neighbor in T. Then $T_2 + x_4 + z_2$ is a Z-attachment, and $G[T \cup T_1 + z_1]$ contains a D_7, a contradiction.

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